

QUASI-PERIODIC AND PERIODIC SOLUTIONS FOR VECTOR NONLINEAR SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider quasi-periodic and periodic (cnoidal) wave solutions of a set of n -component vector nonlinear Schrödinger equations (VNLSE). In a biased photorefractive crystal with a drift mechanism of nonlinear response and Kerr-type nonlinearity, n component nonlinear Schrödinger equations can be used to model self-trapped mutually incoherent wave packets. These equations also model pulse-pulse interactions in wavelength-division-multiplexed channels of optical fibre transmission systems. Quasi-periodic wave solutions for the VNLSE in terms of n -dimensional Kleinian functions are presented. Periodic solutions in terms of Hermite polynomials and generalized Hermite polynomials for n -component nonlinear Schrödinger equations are found.

1. INTRODUCTION

We consider the system of coupled nonlinear Schrödinger equations

$$i\frac{\partial}{\partial t}Q_j + s\frac{\partial^2}{\partial x^2}Q_j + \sigma\left(\sum_{k=1}^n |Q_k|^2\right)Q_j = 0, \quad j = 1, \dots, n, \quad (1.1)$$

where $s = \pm 1$, $\sigma = \pm 1$. These equations are important for a number of physical applications. For example, for photorefractive media with a drift mechanism of nonlinear response, a good approximation describing the propagation of n self-trapped mutually incoherent wave packets is the set of equations for a Kerr-type nonlinearity [29]

$$i\frac{\partial}{\partial z'}\tilde{Q}_j + \frac{1}{2}\frac{\partial^2}{\partial x'^2}\tilde{Q}_j + \alpha\delta\eta\tilde{Q}_j = 0, \quad j = 1, \dots, n, \quad (1.2)$$

where \tilde{Q}_j denotes the j th component of the beam, α is a coefficient representing the strength of nonlinearity, z' and x' are the coordinate

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along the direction of propagation and transverse coordinate respectively. The change in refractive index profile η created by all the incoherent components in the light beam is defined by

$$\delta\eta = \sum_{k=1}^n |\tilde{Q}_k|^2. \quad (1.3)$$

Inserting (1.3) in (1.2) and renormalising the variables as $\tilde{Q}_j = Q_j/\sqrt{2\alpha}$, $z' = 2t$, $x' = x$ we obtain the vector nonlinear Schrödinger equation (1.1). Stability, localization, and soliton asymptotics of multicomponent photorefractive cnoidal waves are discussed in [35]. New solutions in explicit form are presented here for the case $n = 3$. The pulse-pulse collision between wavelength-division-multiplexed channels of optical fibre transmission systems are described with equations of VNLSE type (1.1) [33, 24].

Quasi-periodic solutions in terms of N -phase theta functions for the Manakov system [30], which is the system (1.1) with $n = 2$, are derived in [1], while a series of special solutions are given in [3, 36, 37, 38]. The authors have already discussed quasi-periodic and periodic solutions associated with Lamé and Treibich-Verdier potentials for the Manakov system in the framework of a special ansatz [12]. General quasi-periodic solutions in terms of N -phase theta functions of the VNLSE are derived in [10, 28] and for matrix nonlinear Schrödinger equations in [46]. We also mention the method of constructing elliptic finite-gap solutions of the stationary KdV and AKNS hierarchy, based on a theorem due to Picard, proposed in [19, 21, 20], as well the method developed by Smirnov in a series of publications: the review paper [40] and [41, 42].

In the present paper, we investigate (1.1) by introducing a special ansatz to analyse hyperelliptic and elliptic solutions of the VNLSE.

The paper is organised as follows. In the Section 2 we construct the Lax representation of the finite dimensional system obtained after substitution of the ansatz and develop a hyperelliptic curve, which are associated with the system. In Section 3 we present the integration of the system in terms of *Kleinian hyperelliptic functions*, which give quasi-periodic solutions. Recently this realization of Abelian functions was discussed in [7, 8, 15, 9]. In Section 4 we show how these quasi-period solutions reduce in special cases to give periodic solutions of (1.1) in terms of elliptic functions. We explain also in Section 4 the reduction of Kleinian hyperelliptic functions to Hermite and generalized Hermite polynomials.

We seek solution of (1.1) in the following form (e.g. Porubov & Parker [37])

$$Q_j = q_j(\zeta) e^{i\Theta_j}, \quad j = 1, \dots, n, \quad (1.4)$$

where $\zeta = x - ct$, $\Theta_j = \Theta_j(\zeta, t)$, with q_j, Θ_j real. Substituting (1.4) into (1.1) and separating real and imaginary parts by supposing that the functions $\Theta_j, j = 1, \dots, n$ behave as

$$\Theta_j = \frac{1}{2}scx + (a_j - \frac{1}{4}sc^2)t - sC_j \int_0^\zeta \frac{d\zeta'}{q_j(\zeta')^2} + \Theta_{j0},$$

we obtain the system ($\sigma = s = \pm 1$) [47, 26, 43, 27]

$$\frac{d^2}{dz^2}q_j + \left(\sum_{k=1}^n \frac{\sigma}{s} q_k^2 - \frac{a_j}{s} \right) q_j - \frac{C_j^2}{q_j^3} = 0, \quad k, j = 1, \dots, n, \quad (1.5)$$

where $C_j, j = 1, \dots, n$ are free parameters and Θ_{j0} are constants. These equations describe the integrable case of motion of a particle in a quartic potential perturbed with inverse squared potential, which is separable in ellipsoidal coordinates [47].

2. LAX REPRESENTATION

The system (1.5), with $s = 1, \sigma = 1$ is a completely integrable Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{4} \left(\sum_{i=1}^n q_i^2 \right)^2 - \frac{1}{2} \sum_{i=1}^n a_i q_i^2 + \frac{1}{2} \sum_{i=1}^n \frac{C_i^2}{q_i^2}, \quad (2.1)$$

where the variables $(q_i, p_i), i = 1, \dots, n$, $p_i(\zeta) = dq_i(\zeta)/d\zeta$, are the canonically conjugated variables with respect to the standard Poisson bracket, $\{\cdot; \cdot\}$.

This system has the Lax representation [27]

$$\begin{aligned} \frac{dL(\lambda)}{d\zeta} &= [M(\lambda), L(\lambda)], \\ L(\lambda) &= \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q(\lambda) & 0 \end{pmatrix}, \end{aligned} \quad (2.2)$$

which is equivalent to (1.5), where $U(\lambda), W(\lambda), Q(\lambda)$ have the form

$$\begin{aligned} U(\zeta, \lambda) &= -a(\lambda) \left(1 + \frac{1}{2} \sum_{i=1}^n \frac{q_i^2}{(\lambda - a_i)} \right), \quad V(\zeta, \lambda) = -\frac{1}{2} \frac{dU(\zeta, \lambda)}{d\zeta} \\ W(\zeta, \lambda) &= a(\lambda) \left(-\lambda + \frac{1}{2} \sum_{i=1}^n q_i^2 + \frac{1}{2} \sum_{i=1}^n \frac{1}{\lambda - a_i} \left(p_i^2 + \frac{C_i^2}{q_i^2} \right) \right), \\ Q(\zeta, \lambda) &= \lambda - \sum_{i=1}^n q_i^2. \end{aligned} \quad (2.3)$$

The Lax representation yields the hyperelliptic curve $K = (\nu, \lambda)$

$$\det(L(\lambda) - \frac{1}{2}\nu\mathbf{1}_2) = 0, \quad (2.4)$$

where $\mathbf{1}_2$ is the 2×2 unit matrix. The moduli of the curve (2.4) generate the integrals of motion $H, F^{(i)} = H_i + I_i, i = 1, \dots, n$,

$$\nu^2 = V^2(\zeta, \lambda) + U(\zeta, \lambda)W(\zeta, \lambda). \quad (2.5)$$

The curve (2.5) can be written in canonical form as

$$\nu^2 = 4 \prod_{j=0}^{2n} (\lambda - \lambda_j), \quad (2.6)$$

where $\lambda_j \neq \lambda_k$ are branching points.

From (2.5) and explicit expressions for $U(\zeta, \lambda), V(\zeta, \lambda), W(\zeta, \lambda)$ we obtain

$$\nu^2 = a(\lambda)^2 \left(\lambda - \sum_{i=1}^n \frac{H_i}{\lambda - a_i} - \frac{1}{4} \sum_{i=1}^n \frac{J_i^2}{(\lambda - a_i)^2} + \sum_{i=1}^n \frac{I_i}{\lambda - a_i} \right), \quad (2.7)$$

where

$$I_i = \frac{1}{4} \sum_{k \neq i} \frac{1}{a_i - a_k} \left((q_i p_k - q_k p_i)^2 - \frac{C_i^2 q_k}{q_i^2} - \frac{C_k^2 q_i}{q_k^2} \right),$$

$$H_i = \frac{1}{2} p_i^2 - \frac{1}{2} a_i q_i^2 + \frac{1}{4} q_i^2 \left(\sum_{k=1}^n q_k^2 \right) + \frac{1}{2} \frac{C_i^2}{q_i^2},$$

$$J_i = 2C_i, \quad a(\lambda) = \prod_{i=1}^n (\lambda - a_i),$$

and $\sum_{i=1}^n H_i$ is the Hamiltonian. The parameters C_i are linked with the coordinates of the points $(a_i, \nu(a_i))$ by the formula

$$C_i^2 = -\frac{\nu(a_i)^2}{\prod_{k \neq i} (a_i - a_k)}, \quad (2.8)$$

where $i = 1, \dots, n$. Let us define new coordinates $\mu_i, i = 1, \dots, n$ as zeros of the entry $U(\lambda)$ in the Lax operator. Then

$$q_i^2 = 2 \frac{\prod_{j=1}^n (a_i - \mu_j)}{\prod_{k \neq i}^n (a_i - a_k)}, \quad (2.9)$$

where $i = 1, \dots, n$. The definition of $\mu_i, i = 1, \dots, n$ in combination with the Lax representation leads to the equations

$$\nu_i = V(\mu_i) = -\frac{1}{2} \frac{d}{d\zeta} U(\mu_i), \quad i = 1, \dots, n, \quad (2.10)$$

which can be transformed to equations of the form¹

$$u_i = \sum_{k=1}^n \int_{\lambda_k}^{\mu_k} du_i, \quad (2.11)$$

where $du_i, i = 1, \dots, n$ denote independent canonical holomorphic differentials

$$du_i = \lambda^{i-1} \frac{d\lambda}{\nu}, \quad (2.12)$$

and $u_k = d_k, u_n = 2\zeta + b$ with the constants $d_k, b, k = 1, \dots, n-1$ defined by the initial conditions. The integration of the problem then reduces to the solution of the *Jacobi inversion problem* associated with the curve, which consist of the expression of the symmetric functions of $(\mu_i, \nu_i, i = 1, \dots, n)$ as function of n complex variables $(u_i, i = 1, \dots, n)$.

3. EXACT SOLUTIONS IN TERMS OF KLEINIAN HYPERELLIPTIC FUNCTIONS

In this section we give the trajectories of the system under consideration in terms of Kleinian hyperelliptic functions (see, e.g. [4, 8]), associated with the algebraic curve (2.6) of genus n can be also written in the form

$$\nu^2 = 4 \prod_{j=0}^{2n} (\lambda - \lambda_j) = 4\lambda^{2n+1} + \sum_{j=0}^{2n} \alpha_j \lambda^j. \quad (3.1)$$

At all real branching points $\lambda_j \neq \lambda_k$ the closed intervals $[\lambda_{2i-1}, \lambda_{2i}], i = 1, \dots, n$ will be referred further as lacunae [48, 31]. Let us equip the curve with a homology basis $(\mathbf{a}_i; \mathbf{b}_i, i = 1, \dots, n) \in H_1(K, \mathbb{Z})$ and fix

¹In what follows we shall denote the integral bounds by the second coordinate of the curve $K = K(\nu, \lambda)$ (2.7).

the basis in the space of holomorphic differentials. The canonical meromorphic differentials of the second kind $d\mathbf{r}^T = (dr_i, i = 1, \dots, n)$, associated with (2.12), have the form

$$dr_j = \sum_{k=j}^{2n+1-j} (k+1-j) \alpha_{k+1+j} \frac{\lambda^k d\lambda}{4\nu}, \quad j = 1, \dots, n. \quad (3.2)$$

The $n \times n$ matrices of their periods are

$$\begin{aligned} 2\omega &= \left(\oint_{\mathbf{a}_k} du_l \right)_{k,l=1,\dots,n}, & 2\omega' &= \left(\oint_{\mathbf{b}_k} du_l \right)_{k,l=1,\dots,n}, \\ 2\eta &= \left(- \oint_{\mathbf{a}_k} dr_l \right)_{k,l=1,\dots,n}, & 2\eta' &= \left(- \oint_{\mathbf{b}_k} dr_l \right)_{k,l=1,\dots,n}. \end{aligned}$$

Define the Abel map $K^g \rightarrow \mathbb{C}^g / (2\omega \oplus 2\omega') = \text{Jac}(K)$ of symmetrized product of K to the Jacobi variety whose coordinates $\mathbf{u}^T = (u_1, \dots, u_g)$ are given by the formula (2.11).

The fundamental σ function in this case is a natural generalization of the Weierstrass elliptic σ function and is defined as follows

$$\begin{aligned} \sigma(\mathbf{u}) &= \frac{\sqrt{\pi^n}}{\sqrt{\det(2\omega)}} \frac{\epsilon}{\sqrt[4]{\prod_{0 \leq i < j \leq 2n} (\lambda_i - \lambda_j)}} \\ &\quad \times \exp \{ \mathbf{u}^T \eta (2\omega)^{-1} \mathbf{u} \} \theta[\varepsilon]((2\omega)^{-1} \mathbf{u} | \omega' \omega^{-1}), \end{aligned}$$

where $\epsilon^8 = 1$, $\tau = \omega' \omega^{-1}$ and $\theta[\varepsilon](\mathbf{v} | \tau)$ is the θ -function,

$$\theta[\varepsilon](\mathbf{v} | \tau) = \sum_{\mathbf{m} \in \mathbb{Z}^n} \exp i\pi \{ (\mathbf{m} + \varepsilon)^T \tau (\mathbf{m} + \varepsilon) + 2(\mathbf{v} + \varepsilon')^T (\mathbf{m} + \varepsilon) \},$$

with the half integer characteristic

$$[\varepsilon] = \begin{bmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ \varepsilon'_1 & \dots & \varepsilon'_n \end{bmatrix}, \quad \forall \varepsilon_i, \varepsilon_j = \frac{1}{2} \text{ or } 0.$$

The characteristic $[\varepsilon]$ is chosen in such the way, that it is the characteristic of the point \mathbf{e} at which the θ -function $\theta[0](\mathbf{e})$ vanishes to the order $\lfloor \frac{n+1}{2} \rfloor$. According to the vanishing Riemann theorem and Clifford theorem the Jacobi variety $\text{Jac}(K)$ of the hyperelliptic curve of genus n always has such the point (see e.g. [17])

The $2n \times 2n$ period matrix, $\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}$ satisfies the generalized Legendre relation

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^T \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} = -\frac{i\pi}{2}. \quad (3.3)$$

This relation (3.3) allows us to define a multidimensional σ -function, which has the following properties

- The σ -function possesses the following periodicity property: if

$$\mathbf{E}(\mathbf{m}, \mathbf{m}') = \eta \mathbf{m} + \eta' \mathbf{m}', \quad \text{and} \quad \mathbf{\Omega}(\mathbf{m}, \mathbf{m}') = \omega \mathbf{m} + \omega' \mathbf{m}',$$

where $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^n$, then

$$\begin{aligned} \sigma[\varepsilon](\mathbf{z} + 2\mathbf{\Omega}(\mathbf{m}, \mathbf{m}'), \omega, \omega') &= \exp\{2\mathbf{E}^T(\mathbf{m}, \mathbf{m}')(\mathbf{z} + \mathbf{\Omega}(\mathbf{m}, \mathbf{m}'))\} \\ &\times \exp\{-\pi i \mathbf{m}^T \mathbf{m}' - 2\pi i \varepsilon^T \mathbf{m}'\} \sigma[\varepsilon](\mathbf{z}, \omega, \omega'). \end{aligned}$$

- As a modular function, the Kleinian σ -function is invariant under the transformation of the symplectic group, which represents an important characteristic feature.

We introduce the following notation. Kleinian ζ and \wp -functions are defined as logarithmic derivatives of the fundamental σ -function

$$\begin{aligned} \zeta_i(\mathbf{u}) &= \frac{\partial \ln \sigma(\mathbf{u})}{\partial u_i}, \quad i = 1, \dots, n; \\ \wp_{ij}(\mathbf{u}) &= -\frac{\partial^2 \ln \sigma(\mathbf{u})}{\partial u_i \partial u_j}, \quad \wp_{ijk}(\mathbf{u}) = -\frac{\partial^3 \ln \sigma(\mathbf{u})}{\partial u_i \partial u_j \partial u_k}, \dots, \\ & \quad i, j, k, \dots = 1, \dots, n. \end{aligned}$$

The functions $\zeta_i(\mathbf{u})$ and $\wp_{ij}(\mathbf{u})$ have the following periodicity properties

$$\begin{aligned} \zeta_i(\mathbf{u} + 2\mathbf{\Omega}(\mathbf{m}, \mathbf{m}')) &= \zeta_i(\mathbf{u}) + 2E_i(\mathbf{m}, \mathbf{m}'), \quad i = 1, \dots, n, \\ \wp_{ij}(\mathbf{u} + 2\mathbf{\Omega}(\mathbf{m}, \mathbf{m}')) &= \wp_{ij}(\mathbf{u}), \quad i, j = 1, \dots, n, \end{aligned}$$

where $E_i(\mathbf{m}, \mathbf{m}')$ is the i -th component of the vector $\mathbf{E}(\mathbf{m}, \mathbf{m}') = \eta \mathbf{m} + \eta' \mathbf{m}'$ and $\mathbf{\Omega}(\mathbf{m}, \mathbf{m}') = \omega \mathbf{m} + \omega' \mathbf{m}'$.

Alternatively the σ function can be defined by its expansion near $\mathbf{u} = 0$. In particular, for small genera we have

$$\sigma(\mathbf{u}) = u_1 + o(\mathbf{u}^3) \quad \text{for } n = 1 \text{ and } 2,$$

$$\sigma(\mathbf{u}) = u_1 u_3 - u_2^2 + o(\mathbf{u}^4) \quad \text{for } n = 3 \text{ and } 4,$$

$$\sigma(\mathbf{u}) = -u_3^3 + 2u_2 u_3 u_4 - u_1 u_4^2 - u_2^2 u_5 + u_1 u_3 u_5 + o(\mathbf{u}^5) \quad \text{for } n = 5 \text{ and } 6.$$

and further terms can be computed with the help of a bilinear differential equation [5].

The principal result of the theory is the formula of Klein, which reads in the case of genus n as follows

$$\sum_{k,l=1}^n \wp_{kl} \left(\int_{\infty}^{\mu} d\mathbf{u} - \sum_{m=1}^n \int_{a_m}^{\mu_m} d\mathbf{u} \right) \mu^{k-1} \mu_i^{l-1}$$

$$= \frac{F(\mu, \mu_i) + 2\nu\nu_i}{4(\mu - \mu_i)^2}, \quad i = 1, \dots, n, \quad (3.4)$$

where

$$F(\mu_1, \mu_2) = \sum_{r=0}^n \mu_1^r \mu_2^r [2\alpha_{2r} + \alpha_{2r+1}(\mu_1 + \mu_2)]. \quad (3.5)$$

By expanding these equalities in the neighbourhood of infinity we obtain the complete set of the relations for the hyperelliptic functions.

The first group of relations represents the solution of the Jacobi inversion problem in the form

$$\lambda^n - \wp_{nn}(\mathbf{u})\lambda^{n-1} - \wp_{n,n-1}(\mathbf{u})\lambda^{n-2} - \dots - \wp_{n1}(\mathbf{u}) = 0. \quad (3.6)$$

Let us introduce a polynomial $\mathcal{F}(\mathbf{u}, \lambda)$ in λ of degree n

$$\mathcal{F} = \lambda^n - \wp_{n,n}(\mathbf{u})\lambda^{n-1} - \wp_{n,n-1}(\mathbf{u})\lambda^{n-2} - \dots - \wp_{n,1}(\mathbf{u}). \quad (3.7)$$

which are solutions of the following nonlinear equation

$$2 \left(\frac{d^2}{du_n^2} \mathcal{F} \right) \mathcal{F} - \left(\frac{d}{du_n} \mathcal{F} \right)^2 - 4(\lambda + \mathbf{U})\mathcal{F}^2 + \nu^2 = 0, \quad (3.8)$$

where $\mathbf{U} = 2\wp_{nn} + \frac{1}{4}\alpha_{2n}$. It is possible to show that the following equations are valid

$$\wp_{nnni} = (6\wp_{nn} + \alpha_{2n})\wp_{ni} + 6\wp_{n,i-1} - 2\wp_{n-1,i} + \frac{1}{2}\delta_{ni}\alpha_{2n-1}, \quad (3.9)$$

for $i = 1, \dots, n$. These equations can be identified with the KdV hierarchy with "time variables" $(t_1, t_2, \dots) = (u_n, u_{n-1}, \dots) = (x, t, \dots)$,

$$\mathcal{X}_{k+1}[\mathbf{U}] = \mathcal{R}\mathcal{X}_k[\mathbf{U}], \quad (3.10)$$

where $\mathcal{R} = \partial_x^2 - \mathbf{U} + c - \frac{1}{2}\mathbf{U}_x\partial^{-1}$, and $c = \alpha_{2n}/12$ is the Lenard recursion operator. The first two equations from the hierarchy are

$$\mathbf{U}_{t_1} = \mathbf{U}_x, \quad \mathbf{U}_{t_2} = \frac{1}{2}(\mathbf{U}_{xxx} - 6\mathbf{U}_x\mathbf{U}), \quad (3.11)$$

the second equation is the KdV equation, which is obtained from (3.9) with $i = n$ as the result of differentiation by $x = u_n$ and setting $\mathbf{U} = 2\wp_{nn} + \frac{1}{6}\alpha_{2n}$. The next group of equations are obtained from (3.7,3.8)

and for simplicity we present results only for genus $n = 3$

$$\begin{aligned}
\wp_{333}^2 &= 4\wp_{33}^3 + \alpha_6\wp_{33}^2 + 4\wp_{23}\wp_{33} + \alpha_5\wp_{33} + 4\wp_{22} - 4\wp_{13} + \alpha_4, \\
\wp_{233}^2 &= 4\wp_{23}^2\wp_{33} + \alpha_6\wp_{23}^2 - 4\wp_{22}\wp_{23} + 8\wp_{13}\wp_{23} + 4\wp_{11} + \alpha_2, \\
\wp_{133}^2 &= 4\wp_{13}^2\wp_{33} + \alpha_6\wp_{13}^2 - 4\wp_{12}\wp_{13} + \alpha_0, \\
\wp_{233}\wp_{333} &= 4\wp_{33}^2\wp_{23} + \alpha_6\wp_{23}\wp_{33} - 2\wp_{22}\wp_{33}, \\
&\quad + 4\wp_{13}\wp_{33} - 2\wp_{23}^2 + \frac{1}{2}\alpha_5\wp_{23} + 2\wp_{12} + \frac{1}{2}\alpha_3, \\
\wp_{133}\wp_{233} &= 4\wp_{13}\wp_{23}\wp_{33} + \alpha_6\wp_{13}\wp_{23} - 2\wp_{12}\wp_{23} - 2\wp_{13}\wp_{22} + 4\wp_{13}^2 + \frac{1}{2}\alpha_1, \\
\wp_{133}\wp_{333} &= 4\wp_{13}\wp_{33}^2 + \alpha_6\wp_{13}\wp_{33} - 2\wp_{12}\wp_{33} + 2\wp_{13}\wp_{23} + \frac{1}{2}\alpha_5\wp_{13} - 2\wp_{11}.
\end{aligned}$$

Let us introduce finally the *Baker-Akhiezer* function, which in the framework of the formalism developed is expressible in terms of the Kleinian σ -function as follows

$$\Psi(\lambda, \mathbf{u}) = \frac{\sigma\left(\int_{\infty}^{\lambda} d\mathbf{u} - \mathbf{u}\right)}{\sigma(\mathbf{u})} \exp\left\{\int_{\infty}^{\lambda} d\mathbf{r}^T \mathbf{u}\right\}, \quad (3.12)$$

where λ is arbitrary and \mathbf{u} is the Abel image of an arbitrary point $(\nu_1, \mu_1) \times \dots \times (\nu_n, \mu_n)$. It is straightforward to show by direct calculation, using the relations for three and four-index Kleinian \wp -functions, that $\Psi(\lambda, \mathbf{u})$ satisfy the Schrödinger equation

$$\left(\frac{d^2}{du_n^2} - 2\wp_{nn}(\mathbf{u})\right) \Psi(\lambda, \mathbf{u}) = \left(\lambda + \frac{1}{4}\alpha_{2n}\right) \Psi(\lambda, \mathbf{u}) \quad (3.13)$$

for all (ν, μ) .

Now we are in a position to write the solution of the of the system in terms of Kleinian σ -functions and identify the constants in terms of the moduli of the curve. Using (3.7,2.9), the solutions of (1.5) have the following form in terms of Kleinian functions $\wp_{nm}(\mathbf{u})$, $\wp_{n(m-1)}(\mathbf{u}), \dots, \wp_{n1}(\mathbf{u})$, $m = n, \dots, 1$

$$q_i^2 = 2 \frac{a_i^n - \wp_{n,n}(\mathbf{u})a_i^{n-1} - \wp_{n,n-1}(\mathbf{u})a_i^{n-2} - \dots - \wp_{n,1}(\mathbf{u})}{\prod_{k \neq i} (a_i - a_k)},$$

where the vector $\mathbf{u}^T = (d_j, 2\zeta + b)$, $j = 1, \dots, n-1$. Finally, the solutions of the VNLSE (1.1) reads in this case

$$Q_i(x, t) = \sqrt{2 \frac{\mathcal{F}(\mathbf{u}, a_i)}{\prod_{k \neq i}^n (a_i - a_k)}} \exp(\Theta_i), \quad (3.14)$$

where

$$\Theta_j = \left\{ \frac{1}{2}icx + i(a_j - \frac{1}{4}c^2)t - \frac{1}{2}\nu(a_j) \int_0^\zeta \frac{d\zeta'}{\mathcal{F}(\mathbf{u}, a_j)} \right\}, \quad (3.15)$$

$j = 1, \dots, n$, and we have made use of (3.14) and (1.4).

4. ELLIPTIC PERIODIC SOLUTIONS

In this section we develop a method (see also [26, 16, 14, 11]) which allows us to construct periodic solutions of (1.5) in a straightforward way, based on the application of the spectral theory for the Schrödinger equation with elliptic potentials [2, 31]. We start with the formula (2.9) and with the equation for the Baker function $\Psi(\lambda; \mathbf{u})$.

$$\frac{d^2}{dx^2}\Psi(\lambda, \mathbf{u}) - \mathbf{U}\Psi(x, \mathbf{u}) = (\lambda + \frac{\alpha_{2n}}{4})\Psi(\lambda, \mathbf{u}), \quad (4.1)$$

where we identify the potential

$$\mathbf{U} = 2\wp_{nn} + \frac{1}{6}\alpha_{2n}.$$

We assume, without loss of generality, that the associated curve has the property $\alpha_{2n} = 0$. To make this assumption applicable to the initial curve of the system (1.5) derived from the Lax representation, we shift the spectral parameter

$$\lambda \longrightarrow \lambda + \Delta, \quad \Delta = \frac{2}{2n+1} \sum_{i=1}^n a_i. \quad (4.2)$$

Suppose, that \mathbf{U} is a three, four gap Lamé or three, four gap *Treibich-Verdier* potential, which means that

$$\mathbf{U}(x) = 2 \sum_{i=1}^N \wp(x - x_i), \quad (4.3)$$

where $\wp(x)$ is the standard Weierstrass elliptic function with periods $2\omega, 2\omega'$ and x_i takes values from the set $\{0, \omega_1 = \omega, \omega_2 = \omega + \omega', \omega_3 = \omega'\}$. It is known, that the set of such potentials is exhausted by the

following potentials for genus $n = 3, 4$ [44, 40, 21]

$$U(x) = 12\wp(x), \quad (4.4)$$

$$U(x) = 12\wp(x) + 2\wp(x + \omega_i), \quad i = 1, 2, 3, \quad (4.5)$$

$$U(x) = 6\wp(x) + 6\wp(x + \omega_i) + 2\wp(x + \omega_j), \quad i \neq j = 1, 2, 3, \quad (4.6)$$

$$U(x) = 12\wp(x) + 6\wp(x + \omega_i), \quad i = 1, 2, 3, \quad (4.7)$$

$$U(x) = 12\wp(x) + 2 \sum_{i=1}^3 \wp(x + \omega_i), \quad (4.8)$$

$$U(x) = 12\wp(x) + 6 \sum_{i=1}^2 \wp(x + \omega_i) + 2\wp(x + \omega_3), \quad (4.9)$$

$$U(x) = 20\wp(x), \quad (4.10)$$

$$U(x) = 12\wp(x) + 6 \sum_{i=1}^3 \wp(x + \omega_i), \quad (4.11)$$

The potential (4.4), (4.10) is three, four gap Lamé potential respectively; the potentials (4.5,4.7,4.8,4.6,4.11,4.9) are Treibich-Verdier potentials [44, 39, 40].

To display the class of periodic solutions of system (1.5) we introduce the *generalized Hermite polynomial* $\mathcal{F}(x, \lambda)$ by the formula

$$\mathcal{F}(x, \lambda) = \lambda^n - \pi_{n,n}(x)\lambda^{n-1} - \pi_{n,n-1}(x)\lambda^{n-2} - \dots - \pi_{n,1}(x). \quad (4.12)$$

In the particular case of the Lamé potential

$$U(x) = n(n+1)\wp(x), \quad (4.13)$$

the *generalized Hermite polynomial* coincide with the standard Hermite polynomial [45]

$$\mathcal{F}(x, \wp) = \sum_{r=0}^n c_r(\lambda)(\wp(x) - e_2)^{n-r}, \quad (4.14)$$

where the coefficients c_r are solutions of the following recurrence relation

$$\begin{aligned} & 4r\left(n - r + \frac{1}{2}\right)(2n - r + 1)c_r \\ &= (n - r + 1)[12e_2(n - r)(n - r + 2) - 4e_2(n^2 + n - 3) - 4\lambda]c_{r-1} \\ & - 2(n - r + 1)(n - r + 2)(2n - 2r + 3)(e_1 - e_2)(e_2 - e_3)c_{r-2}. \end{aligned}$$

This recurrent formula is satisfied if we set for $r > n + 2$

$$c_{n+1} = c_{n+2} \dots = 0, \quad \text{and normalize } c_0 = 1.$$

The associated Lamé curve is defined by (2.6), where the λ_j are solutions of recurrence relations given in [18], cf. [45]. The original approach

presented in [18] is useful because the Lamé curve appears in factorised form.

The solutions of the system (1.5) are then given as

$$q_i^2(\zeta) = 2 \frac{\mathcal{F}(\zeta, a_i - \Delta)}{\prod_{k \neq i}^n (a_i - a_k)}, \quad i = 1, \dots, n. \quad (4.15)$$

The final formula for the solutions of the system (1.1) then reads

$$Q_i(x, t) = \sqrt{2 \frac{\mathcal{F}(\zeta, a_i - \Delta)}{\prod_{k \neq i}^n (a_i - a_k)}} \exp(\Theta_i), \quad (4.16)$$

where

$$\Theta_j = \left\{ \frac{1}{2} icx + i(a_j - \frac{1}{4}c^2)t - \frac{1}{2}\nu(a_j - \Delta) \int_0^\zeta \frac{d\zeta'}{\mathcal{F}(\zeta', a_j - \Delta)} \right\},$$

and $i = 1, \dots, n$ and we have made use of (4.15) and (1.4).

We shall consider below examples of genus three, four curves, which are associated with the three, four gap elliptic potentials (4.4), (4.10) and (4.5,4.7,4.8,4.6,4.11,4.9).

Consider the potential (4.4) and construct the associated curve [23]

$$\nu^2 = 4\lambda \prod_{i=1}^3 (\lambda^2 - 6e_i\lambda + 45e_i^2 - 15g_2). \quad (4.17)$$

The Hermite polynomial $\mathcal{F}(\wp(x), \lambda)$ associated with the Lamé potential (4.4) has the form

$$\begin{aligned} \mathcal{F}(\wp(x), \lambda) = & \lambda^3 - 6\wp(x)\lambda^2 - 3 \cdot 5(g_2 - 3\wp^2)\lambda - \\ & \frac{3^2 \cdot 5^2}{4}(4\wp^3 - g_2\wp - g_3). \end{aligned} \quad (4.18)$$

Then the finite and real solution of the system (1.5) is given by (4.16) with the Hermite polynomial depending on the argument $x + \omega'$ (the shift in ω' provides the holomorphy of the solution). The solution is real under the choice of the arbitrary constants $a_i, i = 1, \dots, n$ in such a way, that the constants $a_i - \Delta, i = 1, \dots, n$ lie in *different* lacunae. According to (1.4) the constants C_i are then given as

$$C_i^2 = -\frac{\nu(a_i - \Delta)^2}{\prod_{k \neq i}^n (a_i - a_k)},$$

where $i = 1, \dots, n$ and Δ is the shift (4.2) and ν is the coordinate of the curve (4.17).

The Hermite polynomial $\mathcal{F}(\wp(x), \lambda)$ associated with the Lamé potentials can be written in a different form, useful for applications:

$$\mathcal{F}(\wp(x), \lambda) = \sum_{k=0}^n A_k(\lambda) \wp(x)^{n-k}. \quad (4.19)$$

For example for the genus $n = 4$ Lamé potential (4.10) we have

$$\begin{aligned} A_0 &= 11025, & A_1 &= -1575\lambda, & A_2 &= 135\lambda^2 - \frac{6615}{2}g_2 \\ A_3 &= -10\lambda^3 + \frac{1855}{4}\lambda g_2 - 2450g_3 \\ A_4 &= \lambda^4 - \frac{113}{2}\lambda^2 g_2 + \frac{3969}{16}g_2^2 + \frac{1925}{4}\lambda g_3. \end{aligned}$$

For the general Lamé potential the associated curve is defined by

$$\begin{aligned} \nu^2(\lambda) &= -4\lambda A_n^2 - A_n A_{n-1} g_2 + (A_{n-1}^2 - 4A_{n-2} A_n) g_3 \quad (4.20) \\ &= 4 \prod_{j=0}^{2n} (\lambda - \lambda_j). \end{aligned}$$

For the genus four Lamé potential (4.10) we have

$$\nu^2 = \left(\prod_{l=1}^3 (\lambda^2 + 10e_l \lambda - 7(5e_l^2 + g_2)) \right) (\lambda^3 - 52\lambda g_2 + 560g_3).$$

To display the class of periodic solutions of the system (1.5) for genus $n = 3$ we express the *generalized Hermite polynomial* $\mathcal{F}(x, \lambda)$ by the formula

$$\mathcal{F}(x, \lambda) = \lambda^3 - \pi_{33}(x)\lambda^2 - \pi_{23}(x)\lambda - \pi_{13}(x) \quad (4.21)$$

with $\pi_{33}(x)$, $\pi_{23}(x)$ and $\pi_{13}(x)$ given as follows

$$\begin{aligned} \pi_{33}(x) &= \sum_{j=1}^N \wp(x - x_j) + \frac{1}{3} \sum_{j=0}^6 Z_j, \\ \pi_{12}(x) &= -3 \sum_{i<j} \wp(x - x_i) \wp(x - x_j) - \frac{Ng_2}{8} \\ &\quad - \frac{1}{6} \sum_{i<j} \lambda_i \lambda_j + \frac{1}{6} \left(\sum_{j=0}^6 \lambda_j^2 \right) \\ \pi_{13}(x) &= \frac{5}{2} \pi_{33}^3 - \frac{5}{4} \pi_{33;xx} \pi_{33} + \frac{1}{8} \pi_{33} \alpha_5 - \frac{5}{8} \pi_{33;x}^2 + \frac{1}{16} \pi_{33;xxxx} - \frac{1}{8} \alpha_4, \end{aligned}$$

where x_i are half-periods, N is the degree of the cover, Z_j , $j = 0, \dots, 6$ are potential shifts (see for example [16]). Without loss of generality we

assume that $\alpha_6 = \sum_{j=0}^6 \lambda_j = 0$. For conciseness we denote $\wp = \wp(x)$, $\wp(x + \omega_1) = \wp_1$. Then for the Treibich-Verdier potential (4.5) we find

$$\begin{aligned} \pi_{33} &= 6\wp + \wp_1 - e_1 \\ \pi_{23} &= -45\wp^2 - 18\wp\wp_1 + 18\wp e_1 + 3\wp_1 e_1 + \\ &\quad + \frac{7}{4}g_2 + 156e_1^2 \\ \pi_{13} &= 225\wp^3 + 225\wp^2\wp_1 - 90\wp\wp_1 e_1 - \frac{15}{2}\wp_x\wp_{1;x} - \\ &\quad - 225\wp^2 e_1 + \frac{105}{4}\wp g_2 + \frac{5}{4}\wp_1 g_2 - 900\wp e_1^2 - \\ &\quad - 150\wp_1 e_1^2 + \frac{6237}{2}e_2 + 155e_1^3 - \frac{2079}{2}g_2 - \frac{7}{4}g_2 e_1 - \frac{451}{8}g_3, \end{aligned}$$

and the associated Treibich-Verdier curve has the form [40]

$$\begin{aligned} \nu^2 &= 4(\lambda - 5(e_1 - e_2))(\lambda - 5(e_1 - e_3)) \\ &\quad (\lambda^2 + 12\lambda e_1 - 20(e_1 - e_2)(e_1 - e_3)) \\ &\quad (\lambda^3 + 3e_1\lambda^2 + (-141e_1^2 - 4g_2) + 45e_1(e_1 - 4e_2))(5e_1 + 4e_2). \end{aligned}$$

The full list of generalized Hermite polynomials for other Treibich-Verdier potentials will be published elsewhere.

5. CONCLUSIONS

In this paper we have described a family of elliptic solutions for the vector nonlinear Schrödinger equations using a Lax pair method and the general method of reduction of Abelian functions to elliptic functions. Our approach is systematic in the sense that special solutions (periodic, soliton, etc.) are obtained in a unified way.

In fibre optics applications, periodic and quasi-periodic waves are of interest in optical transmission systems.

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