THE HYPERELLIPTIC ζ-FUNCTION AND THE INTEGRABLE MASSIVE THIRRING MODEL

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Abstract. We provide a treatment of algebro-geometric solutions of the classical massive Thirring system in the framework of the Weierstrass–Klein theory of hyperelliptic functions. We show that the equations of this model generate the characteristic relations of hyperelliptic theory of *even* hyperelliptic curves, the same role that the KdV equation plays for *odd* hyperelliptic curves. We also consider the soliton limit of the solution obtained and derive the Kuznetsov–Mikhailov soliton as the limit.

1. Introduction

The Thirring model, as one of the most remarkable solvable field theory models, has attracted considerable interest since its introduction in 1958, [46]. In $(1+1)$-dimensions the massive classical Thirring model is described by the equations

$$\begin{aligned} -iu_x + 2v + 2|v|^2 u &= 0, \\ -iv_t + 2u + 2|u|^2 v &= 0. \end{aligned} \tag{1.1}$$

The complete integrability of the system (1.1) in the sense of soliton theory was first established by Mikhailov [40] in 1976. This started a comprehensive study of its properties, and an extensive bibliography can be found in, e.g., [27].

The algebro-geometric integration of the model was independently established in 1978 by Date [23] and Holod and Prikarpatsky [33]; it is remarkable that none of these papers completed the integration and both left some quadratures in the exponential. The complete θ-function expression was given in 1984 by Bikbaev [8], however his formulas contained some inaccuracies. Recently Wisse [51] considered algebro-geometric solutions of (1.1), but without deriving explicit formulas.

The problem was recently revisited by Enolskii, Gesztesy, and Holden [27], who derived explicit θ-function solutions in a form close to that given by Bikbaev. The approach used in [27] is based on the Riccati-type equation associated with the Thirring model. This technique is described in detail for various integrable equations in the forthcoming monograph [30]. This book also contains a comprehensive bibliography on the integration of the Thirring model within the inverse scattering method, the application of Bäcklund transform, as well as connections with other integrable equations.

In this paper we continue the investigation of [27] in the framework of the Klein–Weierstrass theory of hyperelliptic σ-functions. The systematic study of the σ-function, which may be traced to papers of Klein [35, 36], was an alternative to the development by Weierstrass [47, 48] (the hyperelliptic generalization of the Jacobi elliptic functions sn, cn, dn) and the purely θ-function theory approach by Göpel [31] and Rosenhain [45] for genus two, generalized further by Riemann. The approach using σ-functions was developed by Burkhardt [21], Wiltheiss [50], Bolza

Date: October 4, 2002.

The research was supported in part by the Research Council of Norway. In addition the authors are grateful for the hospitality of the Newton Institute, Cambridge, where the final version of the paper was developed. The research of the first two authors was supported in part by ESPRC grant GR/R2336/01, and by the EU LOCNET Network HPRN-CT-1999-00163.

[9, 10, 11, 12], Baker [2, 3, 4, 5], and others. Recently this subject was revisited in series of papers of Buchstaber, Enolskii, and Leykin [14, 15, 16, 17, 18], see also [26].

It was shown in particular in the last set of papers, that in the case of an odd hyperelliptic curve, i.e., a curve with a branch point at infinity, the KdV-type hierarchy appears in natural way among the differential relation between Kleinian $\wp$-functions. These functions are defined as second logarithmic derivatives of hyperelliptic σ-functions. Therefore the KdV-hierarchy plays the the role of defining relations of the theory of hyperelliptic functions, while the Kleinian $\wp$-functions appear to be convenient coordinates to describe completely integrable systems.

The aim of this paper is to show that the Thirring model (1.1) represents, in the same way, the master relations for Abelian functions of even hyperelliptic curves, i.e., curves without branch points at infinity. Furthermore, the natural coordinates for these relations are the Kleinian ζ-functions, which are defined as the logarithmic derivatives of the hyperelliptic σ-function.

The paper is organized as follows. We recall in Section 2 the Kleinian realization of hyperelliptic Abelian functions. It is based on the algebraic expression for the symmetric bi-differential with the only pole along the diagonal, as introduced in [35, 36]. This normalized differential is called the Bergmann kernel in the modern literature. In this section we also introduce the hyperelliptic σ-function as a θ-function with exponential multiplier, which provides the invariance of this function with respect to the symplectic group. Section 3 is devoted to the addition theorem for hyperelliptic functions, following Klein [36] and Bolza [9]. A more general form of this theorem is given in Fay [29] and it includes the Fay trisecant relations in a particular case. In the context of this paper the addition theorem is used to derive relations between Kleinian ζ-functions and to solve the Jacobi inversion problem for even hyperelliptic curves. In the Section 4 we give the θ-functional solution of the Thirring model obtained in [27] by another method and show that the Thirring equations follows from the relations between hyperelliptic ζ-functions derived in the previous section. In Section 5 the soliton limit of the solution obtained is derived and the original Kuznetsov–Mikhailov soliton formula [37] is obtained.

To conclude this introduction we remark that the Klein–Weierstrass theory of Abelian functions now attracts much interest. Applications of the theory of the σ-function were given in the theory of complex multiplication [32, 41], blow-up formulas in Donaldson–Witten theory [24], theory of solitons [15, 38, 22, 25], the theory of multi-dimensional Schrödinger equations with Abelian potential [13], addition theorems for Abelian functions in determinant form [42, 43], elasticity theory [39], Lie algebras associated with σ-functions and versal deformations [19, 20].

2. Hyperelliptic σ-function

In this Section we recall basic definitions related to the Riemann surface of a hyperelliptic curve. We shall introduce differentials of the first, second and third kind, Kleinian bi-differentials, as well as θ- and σ-functions.

2.1. **Differentials of a hyperelliptic curve.** Let the hyperelliptic curve V of genus g be given by

$$V = \{P = (x, y) \mid y^2 = R(x)\}, \quad R(x) = \sum_{k=0}^{2g+1} \lambda_k x^k = \lambda_{2g+1} \prod_{k=0}^{2g+1} (x - E_k). \tag{2.1}$$

We compactify V by adding two distinct points, $P_{\infty_\pm} = (\infty, \pm\infty)$, at infinity. The canonical homology basis for V is denoted by $\{\mathfrak{a}_1, \dots, \mathfrak{a}_g; \mathfrak{b}_1, \dots, \mathfrak{b}_g\}$, see Fig. 1.

The Abelian differentials associated with V, which are differentials of the first, second and third kind, are described by the following classical theorem.

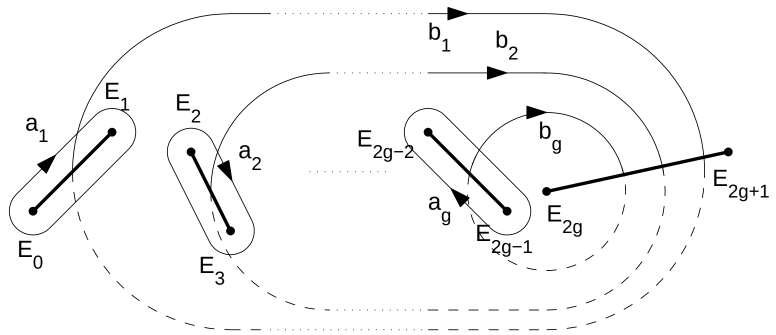

FIGURE 1. Basis of cycles of the hyperelliptic curve V of genus g with branching points $E_0, \dots, E_{2g+1}$.

Theorem 2.1 (see, e.g., [2]). *Let V be a hyperelliptic curve of genus g and fix the set of canonical holomorphic differentials,* $\mathrm{d}\mathbf{u}^T = (\mathrm{d}u_1, \dots, \mathrm{d}u_g)$ *in the form*

$$\mathrm{d}u_k(P) = \frac{x^{k-1}}{y}\mathrm{d}x,\ k = 1, \dots, g,\ P = (x, y). \tag{2.2}$$

Choose the associated set of differentials of the second kind, $\mathrm{d}\mathbf{r}^T = (\mathrm{d}r_1, \dots, \mathrm{d}r_g)$ *in the form*

$$\mathrm{d}r_i(P) = \frac{\mathrm{d}x}{4y}\sum_{k=i}^{2g+1-i}(k+1-i)\lambda_{k+1+i}x^k, \quad i = 1, \dots, n. \tag{2.3}$$

Then the two-differential $\mathrm{d}\Omega(P, Q)$*, given by the formula*

$$\mathrm{d}\Omega(P,Q) = \mathrm{d}z\frac{\mathrm{d}x}{2y}\frac{\partial}{\partial z}\frac{y+w}{x-z} + \sum_{i=1}^{g}\mathrm{d}r_i(Q)\mathrm{d}u_i(P) \tag{2.4}$$

$$= \frac{2yw + F(x,z)}{(x-z)^2}\frac{\mathrm{d}x}{2y}\frac{\mathrm{d}z}{2w}, \quad P = (x,y),\ Q = (z,w), \tag{2.5}$$

where

$$F(x,z) = 2\lambda_{2g+2}x^{g+1}z^{g+1} + \sum_{i=0}^{g}x^iz^i(2\lambda_{2i} + \lambda_{2i+1}(x+z)) \tag{2.6}$$

is symmetric, $\mathrm{d}\Omega(P,Q) = \mathrm{d}\Omega(Q,P)$ *and has a unique pole of second order along the diagonal. The differential* $\mathrm{d}\Omega(P,Q)$ *gives Klein's commutative integral of the third kind*

$$\mathcal{K}(P,P';Q,Q') = \int_P^{P'}\int_Q^{Q'}\frac{2yw + F(x,z)}{4(x-z)^2yw}\mathrm{d}x\mathrm{d}z. \tag{2.7}$$

The set of periods of the differentials (2.2), (2.3)

$$2\omega = \left(\oint_{\mathfrak{a}_k}\mathrm{d}u_l\right)_{k,l=1,\dots,g}, \qquad 2\omega' = \left(\oint_{\mathfrak{b}_k}\mathrm{d}u_l\right)_{k,l=1,\dots,g},$$

$$-2\eta = \left(\oint_{\mathfrak{a}_k}\mathrm{d}r_l\right)_{k,l=1,\dots,g}, \qquad -2\eta' = \left(\oint_{\mathfrak{b}_k}\mathrm{d}r_l\right)_{k,l=1,\dots,g},$$

(where ω, ω', η, η' are $g \times g$-matrices, ω is necessarily non-degenerate) and the cycles $\mathfrak{a}_i, \mathfrak{b}_i$, $i = 1, \dots, g$ constitute the canonical dissection of the Riemann surface of V. The matrices satisfy the conditions

$$\omega'\omega^T - \omega{\omega'}^T = 0, \quad \eta'\omega^T - \eta{\omega'}^T = -\frac{i\pi}{2}1_g, \quad \eta'\eta^T - \eta{\eta'}^T = 0. \tag{2.8}$$

We define the Jacobi variety $\mathrm{Jac}(V)$ of V by

$$\mathrm{Jac}(V) = \mathbb{C}^g/(2\omega \oplus 2\omega'). \tag{2.9}$$

The Abel map maps the symmetric power $\mathrm{symm}^{g+n}V$, $n \in \mathbb{N}$, onto $\mathrm{Jac}(V)$ by the formula

$$w_k = \sum_{j=1}^{g+n} \int_{Q_0}^{P_j} \mathrm{d}u_k, \quad n \geq 0,$$

where $Q_0 \in V$ is arbitrary and $P_1, \dots, P_{g+n}$ is a non-special divisor of degree $g+n$.

2.2. **Hyperelliptic θ and σ-functions.** Next we construct the 2^{2g} Kleinian σ-functions with characteristics $[\varepsilon]$ of g arguments $\boldsymbol{w} = (w_1, \dots, w_g)^T \in \mathrm{symm}^{g+n}V$ by

$$\sigma[\varepsilon](\boldsymbol{u}) = C[\varepsilon] \exp\left\{\boldsymbol{u}\eta(2\omega)^{-1}\boldsymbol{u}\right\} \theta[\varepsilon]((2\omega)^{-1}\boldsymbol{u} - \boldsymbol{K}_{Q_0}), \tag{2.10}$$

Here $C[\varepsilon]$ is a constant, $\boldsymbol{K}_{Q_0}$ is the vector of Riemann constants with base point Q_0, $\theta[\varepsilon](\boldsymbol{v})$ is the standard θ-function of the curve V with characteristic

$$[\varepsilon] = \left[\begin{array}{c} \boldsymbol{\varepsilon}'^T \\ \boldsymbol{\varepsilon}^T \end{array}\right] = \left[\begin{array}{ccc} \varepsilon'_1 & \dots & \varepsilon'_g \\ \varepsilon_1 & \dots & \varepsilon_g \end{array}\right] \in \mathbb{R}^{2g}$$

and the τ-matrix is given as $\tau = \omega'\omega^{-1}$, viz.,

$$\theta[\varepsilon](\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp\left\{\pi i\big((\boldsymbol{m}+\varepsilon')^T\tau(\boldsymbol{m}+\varepsilon') + 2(\boldsymbol{v}+\varepsilon)^T(\boldsymbol{m}+\varepsilon')\big)\right\}.$$

The θ-function has the following transformation properties at the shift on a period:

$$\begin{aligned} &\theta\left[\varepsilon\right](\boldsymbol{v}+\boldsymbol{p}+\tau\boldsymbol{q}|\tau) \\ &= \exp\left(-\imath\pi\boldsymbol{p}^T\tau\boldsymbol{p} - 2\imath\boldsymbol{p}^T\boldsymbol{v} + 2\imath\pi(\boldsymbol{p}^T\boldsymbol{\varepsilon}' - \boldsymbol{q}^T\boldsymbol{\varepsilon}\right)\theta\left[\epsilon\right](\boldsymbol{v}|\tau), \end{aligned} \tag{2.11}$$

where $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{Z}^g$.

In what follows we choose the base point Q_0 of the Abel map at the branching point

$$Q_0 = (0, 0).$$

We shall consider only half-integer characteristics, ε_j, ε'_j equal to 0 or 1/2. There exist 4^g such transcendental characteristics, and they are in one-to-one correspondence with the 4^g algebraic characteristics

$$\phi_{[\varepsilon]} = \prod_{k=1}^{g+1-2m}(x - E_{i_k}), \quad \psi_{[\varepsilon]} = \prod_{k=1}^{g+1+2m}(x - E_{j_k}) \tag{2.12}$$

for integers $0 \leq m \leq \left[\frac{g+1}{2}\right]$ and depend on the canonical dissection of the Riemann surface. We introduce the aforementioned correspondence between partitions (2.12) and characteristics by the formula

$$\boldsymbol{\varepsilon} + \tau\boldsymbol{\varepsilon}' = \sum_{k=1}^{g+1-2m} \int_{Q_0}^{(E_{i_k},0)} \mathrm{d}\mathbf{u} + \boldsymbol{K}_{Q_0}.$$

The characteristics are odd or even whenever m is odd or even. Even characteristics with $m = 0$ and odd characteristics with $m = 1$ are called *non-singular*; the characteristics with $m > 1$ are *singular*. The integer m shows the order of vanishing of the θ- or σ-function and according to the Clifford theorem is less than or equal to the positive integer $\left[\frac{g+1}{2}\right]$ (see, e.g., [28]).

Choose the canonical of cycles in such a way that the algebraic characteristic, which we will call the *fundamental characteristic* is [0], that is, (see Fig. 1)

$$\phi_{[0]} = \prod_{k=0}^{g+1}(x - E_{2k}), \quad \psi_{[0]} = \prod_{k=1}^{g+1}(x - E_{2k-1}).$$

The above choice of fundamental characteristics defines the vector of Riemann constants as (see [28, p. 305])

$$\boldsymbol{K}_{Q_0} = \sum_{k=0}^{g} \int_{Q_0}^{(E_{2k},0)} \mathrm{d}\mathbf{u}. \tag{2.13}$$

The Kleinian σ-function, which is associated with the fundamental characteristic, is called the *fundamental σ-function.*

The value of the constant $C[\varepsilon]$ in (2.10) is not needed for what follows, but it provides important invariant properties of the Kleinian σ-function — *the fundamental σ-function is invariant under the action of the symplectic group* $\mathrm{Sp}(2g,\mathbb{Z})$. For completeness we shall give the full definition of the fundamental Kleinian σ-function.

Definition 2.2. The hyperelliptic fundamental σ-function is defined by the formula

$$\sigma(\boldsymbol{u}) = \sqrt{\frac{\pi^g}{\det(2\omega)}} \frac{\epsilon}{\sqrt[4]{\prod_{1\le i<j\le 2g+2}(E_i - E_j)}} \exp\left(\boldsymbol{u}^T \eta(2\omega)^{-1}\boldsymbol{u}\right) \theta((2\omega)^{-1}\boldsymbol{u}|\tau), \tag{2.14}$$

where $\epsilon^4 = 1$, and

$$\boldsymbol{u} = \sum_{j=1}^{g} \int_{(E_{2j},0)}^{P_j} \mathrm{d}\mathbf{u}. \tag{2.15}$$

Note that equivalently the Kleinian σ-function can be defined as a power series in u_k, $k = 1,\ldots,g+n$ with coefficients given recursively, see [10, 11, 12].

The Kleinian ζ- and $\wp$-functions are defined as the logarithmic derivatives of the fundamental σ-function

$$\begin{aligned}
\zeta_i(\boldsymbol{u}) &= \frac{\partial \ln\, \sigma(\boldsymbol{u})}{\partial u_i}, \quad i = 1,\ldots,g,\\
\wp_{ij}(\boldsymbol{u}) &= -\frac{\partial^2 \ln\, \sigma(\boldsymbol{u})}{\partial u_i \partial u_j}, \quad \wp_{ijk}(\boldsymbol{u}) = -\frac{\partial^3 \ln\, \sigma(\boldsymbol{u})}{\partial u_i \partial u_i \partial u_k} \quad \text{etc.}, i,j,k = 1,\ldots,g.
\end{aligned}$$

The functions $\zeta_i(\boldsymbol{u})$ and $\wp_{ij}(\boldsymbol{u})$ possess the following periodicity properties

$$\begin{aligned}
\zeta_i(\boldsymbol{u} + 2\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}')) &= \zeta_i(\boldsymbol{u}) + 2E_i(\boldsymbol{m},\boldsymbol{m}'), \quad i = 1,\ldots,n,\\
\wp_{ij}(\boldsymbol{u} + 2\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}')) &= \wp_{ij}(\boldsymbol{u}), \quad i,j = 1,\ldots,n,
\end{aligned}$$

where $E_i(\boldsymbol{m},\boldsymbol{m}')$ is the ith component of the vector $E(\boldsymbol{m},\boldsymbol{m}') = \eta\boldsymbol{m} + \eta'\boldsymbol{m}'$ and $\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}') = \omega\boldsymbol{m} + \omega'\boldsymbol{m}'$.

3. The ζ-formula of Bolza

In this Section we describe the very general addition formula for hyperelliptic σ-functions due to Klein and Bolza. We introduce the Schottky-Klein prime form and derive two equivalent expressions for it. As an application of the addition formula for σ-functions, we derive addition formulae for hyperelliptic ζ-functions. The solution of the Jacobi inversion problem for even hyperelliptic curves will be obtained as a consequence of the addition formulae for ζ-functions. We shall also derive a special relation between ζ-functions which will be used later to solve the Thirring model.

3.1. **The main addition formula.**

The Kleinian σ-function, which is associated with the algebraic characteristic $[\varepsilon]$ given by polynomials $\phi_{[\varepsilon]}(x)$, $\psi_{[\varepsilon]}(x)$ of degrees

$2g+1-2m$ and $2g+1+2m$, respectively, is defined, following Klein [36] and Bolza [9, 10, 11, 12], by

$$\sigma[\varepsilon]\left(\sum_{k=1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{u}\right)=\frac{C[\varepsilon]\mathcal{D}[\varepsilon]\prod_{1\le i,k\le g+n}\mathcal{E}(x_i,\xi_k)}{\prod\limits_{1\le i,k\le g+n}(x_i-\xi_k)\prod\limits_{1\le i<k\le g+n}\mathcal{E}(x_i,x_k)\prod\limits_{1\le i<k\le g+n}\mathcal{E}(\xi_i,\xi_k)}. \tag{3.1}$$

In this formula $P_k=(x_k,y_k)$, $Q_k=(\xi_k,\nu_k)$, $n\ge 0$, and $C[\varepsilon]$ is the previously mentioned constant, $\mathcal{E}(x,\xi)$ is the Schottky–Klein *prime form*,

$$\mathcal{E}(x,\xi)=(x-\xi)\exp\left(\frac{1}{2}\mathcal{K}(P,Q,\bar{P},\bar{Q})\right), \tag{3.2}$$

where $\mathcal{K}(P,Q,\bar{P},\bar{Q})$ is the commutative Kleinian integral of the third kind (2.7) and where $\bar{P}=(x,-y)$ whenever $P=(x,y)$ (similarly for Q). $\mathcal{D}[\epsilon]$ denotes the $2(g+n)\times 2(g+n)$ determinant

$$\mathcal{D}[\varepsilon]=\begin{vmatrix}\mathcal{D}_\phi[\varepsilon](x) & \mathcal{D}_\psi[\varepsilon](x)\\ -\mathcal{D}_\phi[\varepsilon](\xi) & \mathcal{D}_\psi[\varepsilon](\xi)\end{vmatrix}, \tag{3.3}$$

where

$$\mathcal{D}_\phi[\varepsilon](x)=\begin{pmatrix}\sqrt{\phi(x_1)} & x_1\sqrt{\phi(x_1)} & \dots & x_1^{g+n+m-1}\sqrt{\phi(x_1)}\\ \vdots & \vdots & \dots & \vdots\\ \sqrt{\phi(x_{g+n})} & x_{g+n}\sqrt{\phi(x_{g+n})} & \dots & x_{g+n}^{g+n+m-1}\sqrt{\phi(x_{g+n})}\end{pmatrix},$$
$$x=(x_1,\dots,x_{g+n}).$$

The Schottky–Klein prime form $\mathcal{E}(x,z)$, defined by (3.2), is a multivalued function with the following properties. $\mathcal{E}(x,z)=0$ if and only if $(x,y)=(z,w)$ and

$$\lim_{x\to z}\frac{\mathcal{E}(x,z)}{x-z}=1.$$

Furthermore, $\mathcal{E}(x,z)=-\mathcal{E}(z,x)$. For fixed x the multiplier along the $\mathfrak{a}_i$ and $\mathfrak{b}_i$ cycles are $\boldsymbol{\eta}_i^T(\boldsymbol{u}+\boldsymbol{\omega}_i)$ and $\boldsymbol{\eta}_i'^T(\boldsymbol{u}+\boldsymbol{\omega}_i')$, respectively, where $\boldsymbol{\omega}_i$, $\boldsymbol{\omega}_i'$, $\boldsymbol{\eta}_i$, $\boldsymbol{\eta}_i'$ are the ith columns of the corresponding matrices.

We briefly recall the principal ideas in the derivation of (3.1) in the case of a hyperelliptic curve (see, e.g., [2] for more details in a more general case). Let $P_1,\dots,P_{g+n}$, $Q_1,\dots,Q_{g+n}$, $n\ge 0$, are such divisors of degree $g+n$

$$\sigma\left(\sum_{k=1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{u}\right)\not\equiv 0. \tag{3.4}$$

This σ-function can be represented as a function on the symmetric product $\mathrm{symm}^g V$ of V as follows.

Let us consider this function as a function of variable P_1. The θ-function does not vanish identically, and according to the Riemann theorem we can write

$$\theta\left(\sum_{k=1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{u}\right)=\theta\left(\int_{Q_0}^{P_1}\mathrm{d}\mathbf{u}-\sum_{k=1}^{g}\int_{Q_0}^{\widetilde{P}_k}\mathrm{d}\mathbf{u}-\boldsymbol{K}_{Q_0}\right),$$

where $\widetilde{P}_1,\dots,\widetilde{P}_g$ are some points and $\boldsymbol{K}_{Q_0}$ is the vector of Riemann constants (2.13). The last equality yields the congruence

$$\sum_{k=2}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{u}+\sum_{k=1}^{g}\int_{(E_{2k},0)}^{\widetilde{P}_k}\mathrm{d}\mathbf{u}=0.$$

Abel's theorem implies that there exists a meromorphic function $\Delta(x)$, which has zeros $P_2,\ldots,P_{g+n}$, $\widetilde{P}_1,\ldots,\widetilde{P}_g$ and poles $Q_2,\ldots,Q_{g+n}$, $(E_2,0),\ldots,(E_{2g},0)$. Moreover, according to the Riemann–Roch theorem, such a function is uniquely determined.

It is straightforward to check that this definition permits the representation of the σ-function (3.4) in the form

$$\sigma[\varepsilon](\boldsymbol{w}) = C[\varepsilon]\Delta[\varepsilon]\frac{\prod\limits_{i,j=1,\ldots,n}\mathcal{E}(x_i,z_i)}{\prod\limits_{1\le i<j\le n}\mathcal{E}(x_i,x_j)\mathcal{E}(z_i,z_j)}, \tag{3.5}$$

where $\Delta[\varepsilon]$ is a meromorphic function on $\mathrm{symm}^{g+n}V$ and $C[\varepsilon]$ the constant depending on the characteristic and moduli of the curve.

3.2. **The Schottky-Klein prime form.** To complete the definition it remains to construct the Schottky–Klein prime form $\mathcal{E}(x-z)$ explicitly. This can be done as follows, see [2]. Notice that

$$\mathcal{E}(x,z) = \lim_{\substack{x\to x'\\ z\to z'}}\sqrt{-(x-x')(z-z')\ \exp\left(\frac{1}{2}\mathcal{K}(P,P',Q,Q')\right)}. \tag{3.6}$$

The right-hand side of (3.6) can be computed explicitly. To do that we denote zeros and poles of the rational function

$$\phi(x') = \frac{x'-x}{x'-z}$$

of x' on the two-sheeted Riemann surface as (x,y), (x,y_1) and (z,w), (z,w_1), respectively.

Abel's theorem states that

$$\begin{aligned}
&\mathcal{K}((x',y'),(x,y),(z',w'),(z_1,w_1))\\
&\quad=\mathcal{K}((x',y'),(x,y),(z',w'),(z,w)) + \ln\left(\frac{(x'-x)(z'-z)}{(x'-z)(z'-x)}\right)\\
&\quad=\ln\left\{\frac{(x'-x)(z'-z)\theta\left(\int\limits_{(z,w)}^{(x',y')}\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right)\theta\left(\int\limits_{(x,y)}^{(z',w')}\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right)}{(x'-z)(z'-x)\theta\left(\int\limits_{(x',y')}^{(x,y)}\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right)\theta\left(\int\limits_{(z',w')}^{(z,w)}\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right)}\right\},
\end{aligned} \tag{3.7}$$

where $\omega\boldsymbol{n}+\omega'\boldsymbol{n}'$ is a nonsingular odd half-period. Suppose, that (x',y'), (z',w') are not branching points. Then the right-hand side of (3.7) can be written as

$$\ln\frac{1}{-(x-z)^2}\frac{\theta\left(\int\limits_Q^P\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right)\theta\left(\int\limits_Q^P\mathrm{d}\mathbf{v}-\omega\boldsymbol{n}-\omega'\boldsymbol{n}'\right)}{\sqrt{\sum_{i=1}^g\frac{\partial\theta(\omega\boldsymbol{n}+\omega'\boldsymbol{n}')}{\partial u_i}\frac{\mathrm{d}v_i(x)}{\mathrm{d}x}}\sqrt{\sum_{i=1}^g\frac{\partial\theta(\omega\boldsymbol{n}+\omega'\boldsymbol{n}')}{\partial u_i}\frac{\mathrm{d}v_i(z)}{\mathrm{d}z}}}.$$

Taking into account that

$$\begin{aligned}
\theta\left(\int\limits_Q^P\mathrm{d}\mathbf{v}-\omega\boldsymbol{n}-\omega'\boldsymbol{n}'\right) &= \exp\left(i\pi\boldsymbol{n}^T\boldsymbol{n}'+2i\pi\boldsymbol{n}'^T\int\limits_Q^P\mathrm{d}\mathbf{v}\right)\theta\left(\int\limits_Q^P\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right)\\
&= -\exp\left(2i\pi\boldsymbol{n}'^T\int_Q^P\mathrm{d}\mathbf{v}\right)\theta\left(\int\limits_Q^P\mathrm{d}\mathbf{v}+\omega\boldsymbol{n}+\omega'\boldsymbol{n}'\right),
\end{aligned}$$

we obtain two equivalent representations

$$\mathcal{E}(x,z)=\frac{\theta\left(\int_Q^P \mathrm{d}\mathbf{v}-\omega\boldsymbol{n}-\omega'\boldsymbol{n}'\right)\exp\left(i\pi\boldsymbol{n}'^T\int_Q^P \mathrm{d}\mathbf{v}\right)}{\sqrt{\sum_{i=1}^g \frac{\partial\theta(\omega\boldsymbol{n}+\omega'\boldsymbol{n}')}{\partial u_i}\frac{\mathrm{d}v_i(x)}{\mathrm{d}x}}\sqrt{\sum_{i=1}^g \frac{\partial\theta(\omega\boldsymbol{n}+\omega'\boldsymbol{n}')}{\partial u_i}\frac{\mathrm{d}v_i(z)}{\mathrm{d}z}}} \tag{3.8}$$

$$=(x-z)\exp\left(\frac{1}{2}\mathcal{K}(P,\bar{P},Q,\bar{Q})\right), \tag{3.9}$$

where in the representation (3.8) the quantity $\omega\boldsymbol{n}+\omega'\boldsymbol{n}'$ is a nonsingular half-period, and in the representation (3.9) the points $P=(x,y)$ and $\bar{P}=(x,-y)$, and $Q=(z,w)$ and $\bar{Q}=(z,-w)$ belong to different sheets of the Riemann surface V.

Note that the first representation of the Schottky–Klein prime form (3.8) was developed by Fay [29], who, based on this, derived the so-called *trisecant Fay formulas*. In what follows we shall use the second representation (3.9) which was mentioned in [29], but was little used in modern literature until the recent series of publications [14, 15, 16, 17, 18] and also [38, 44].

3.3. **The ζ-formula.** The following result can be deduced from the Klein formula for the hyperelliptic σ-function.

Theorem 3.1 (Bolza [9]). *Let $[\varepsilon]$ be an arbitrary algebraic characteristic and $P_1=(x_1,y_1),\dots,P_{g+n}=(x_{g+n},y_{g+n})$, $Q_1=(\xi_1,\nu_1),\dots,Q_{g+n}=(\xi_{g+n},\nu_{g+n})$, $n\geq 0$, be distinct points on V. Let*

$$\boldsymbol{w}=\sum_{k=1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{u},\quad \boldsymbol{r}=\sum_{k=1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{r}.$$

Then for arbitrary $i=1,\dots,g+n$ we have

$$y_i\frac{\partial\ln\sigma[\varepsilon]}{\partial x_i}=\sum_{k=1}^{g}x_i^{k-1}\int_{P_i}^{P_k}\mathrm{d}r_k+\Sigma_i[\varepsilon](P,Q), \tag{3.10}$$

where

$$\Sigma_i[\varepsilon](P,Q)=y_i\frac{\partial\ln\mathcal{D}[\varepsilon]}{\partial x_i}-\frac{1}{2}\sum_{k=1}^{g+n}\frac{y_i-\nu_k}{x_i-\xi_k}-\frac{1}{2}\sum_{\substack{k=1,\dots,g+n\\k\neq i}}\frac{y_i+y_k}{x_i-x_k}.$$

The Kleinian $\zeta[\varepsilon]$-functions are then given by

$$-\zeta_j\left(\boldsymbol{u}+\sum_{k=g+1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}\mathbf{u}\right)=\sum_{k=1}^{g+n}\int_{Q_k}^{P_k}\mathrm{d}r_j+\mathfrak{R}_j^n(P,Q), \tag{3.11}$$

where the functions $\boldsymbol{\mathfrak{R}}^n=(\mathfrak{R}_1^n,\dots,\mathfrak{R}_g^n)^T$ are given by

$$\boldsymbol{\mathfrak{R}}=V^{-1}\boldsymbol{\Sigma}[\varepsilon]$$

where V is the Vandermonde matrix of order g, viz.,

$$V=\begin{pmatrix}1 & x_1 & \dots & x_1^{g-1}\\ 1 & x_2 & \dots & x_2^{g-1}\\ \vdots & \vdots & \dots & \vdots\\ 1 & x_g & \dots & x_g^{g-1}\end{pmatrix}$$

and $\boldsymbol{\Sigma}[\varepsilon]=(\Sigma[\varepsilon]_1,\dots,\Sigma[\varepsilon]_g)^T$.

Proof. By logarithmic differentiation of (3.1) we get

$$\frac{\partial\ln\sigma[\varepsilon]}{\partial x_i}=\frac{\partial\ln\mathcal{D}[\varepsilon]}{\partial x_i}-\sum_k\frac{1}{x_i-\xi_k}$$

$$+\sum_{k=1}^{g+n}\frac{\partial\ln\mathcal{E}(x_i,\xi_k)}{\partial x_i}-\sum_{\substack{k=1,\ldots,g+n\\k\neq i}}\frac{\partial\ln\mathcal{E}(x_i,\xi_k)}{\partial x_i}. \tag{3.12}$$

To proceed we prove that

$$\frac{\partial\ln\mathcal{E}(x_i,\xi_k)}{\partial x_i}=\frac{y_i+y_k}{2(x_i-x_k)y_i}+\sum_{j=1}^{g+n}\frac{x_i^{j-1}}{y_i}\int_{P_i}^{P_j}\mathrm{d}r_k. \tag{3.13}$$

In fact,

$$\frac{\partial\ln\mathcal{E}(x_i,\xi_k)}{\partial x_i}=\frac{1}{x_i-x_k}-\frac{1}{4}\frac{R'(x_i)}{R(x_i)}+\frac{1}{2}\frac{\partial}{\partial x_i}\mathcal{K}(P_i,\bar{P}_i,Q_i,\bar{Q}_i)$$

and since $\bar{x}_i=x_i,\bar{y}_i=-y_i$,

$$\begin{aligned}\frac{\partial}{\partial x_i}\mathcal{K}(P_i,\bar{P}_i,Q_i,\bar{Q}_i)&=2\int_{P_k}^{P_i}\frac{y_iy-F(x_i,x)}{2(x_i-x)^2y_iy}\mathrm{d}x\\&=\frac{1}{2}\frac{y-y_i}{(x-x_i)y_i}\Big|_{x_k}^{x_i}+\sum_{k=1}^{g+n}\frac{x_i^{k-1}}{y_i}\int_{P_i}^{P_k}\mathrm{d}r_k,\end{aligned}$$

where we used (2.4),(2.5). Since

$$\frac{1}{2}\frac{y-y_i}{(x-x_i)y_i}\Big|_{x=x_i}=\frac{1}{2}\frac{R'(x_i)}{R(x_i)},$$

we get (3.10). The formula (3.11) is derived from (3.10) after transformation of the left-hand side to the form

$$\sum_{j=1}^{g}\zeta_j[\varepsilon](\boldsymbol{w})x_i^{j-1}$$

and solving the first g equations with respect to the ζ_j. □

The formula (3.11) is of great generality. We shall restrict it to the case of the fundamental σ-function, i.e., to the case of zero transcendental characteristic and choose the lower limits as

$$\xi_k=E_{2k-1}\quad\text{at}\quad k\leq n\quad\text{and}\quad\xi_k=E_{2g+1}\quad\text{at}\quad k>n, \tag{3.14}$$

so that

$$\boldsymbol{w}=\boldsymbol{u}+\sum_{i=g+1}^{g+n}\int_{Q_0}^{P_i}\mathrm{d}\mathbf{u},\quad\boldsymbol{u}=\sum_{i=1}^{g}\int_{(E_{2k-1},0)}^{P_i}\mathrm{d}\mathbf{u}.$$

Under these assumptions $\mathcal{D}[\varepsilon]$ is reduced to

$$\Delta_0=\mathrm{const}\prod_{i=1}^{g+n}\phi(x_i)\prod_{1\leq i<k\leq g+n}(x_i-x_k)$$

and

$$\frac{\partial\ln\,\Delta_0}{\partial x_i}=\frac{1}{2}\frac{\phi_0'(x_i)}{\phi_0(x_i)}+\sum_{\substack{k=1,\ldots,g+n\\k\neq i}}\frac{1}{x_i-x_k}.$$

The formula (3.10) is then reduced to the form

$$y_i\frac{\partial\ln\,\sigma}{\partial x_i}=\frac{1}{2}\sum_{\substack{k=1,\ldots,g+n\\k\neq i}}\frac{y_i-y_k}{x_i-x_k}+\sum_{l=1}^{g}x_i^{l-1}\sum_{j=1}^{g+n}\int_{(E_{2j+1},0)}^{P_j}\mathrm{d}r_l. \tag{3.15}$$

Let us solve the first g equations (3.15) with respect to $\zeta_j(\boldsymbol{w})$. The functions $\mathfrak{R}_j^n$ can be expressed in closed form in terms of special symmetric functions, which we shall define below.

Definition 3.2. (i) The umbral derivative [44] $D_s(p(z))$ of a polynomial

$$p(z) = \sum_{k=0}^{g} p_k z^k$$

is given by

$$D_s p(z) = \left(\frac{p(z)}{z^s}\right)_+ = \sum_{k=s}^{g} p_k z^{k-s},$$

where $(\cdot)_+$ means the purely polynomial part.
(ii) Let $\boldsymbol{I}$ be the set of integers $\boldsymbol{I} = \{g+1, g+2, \ldots, g+n\}$. For each subset $\{i_1, \ldots, i_m\} \subset \boldsymbol{I}$, and $m \leq N$ define the polynomial

$$R_{i_1,\ldots,i_m}(z) = \frac{\prod_{l=1}^{g+n}(z-x_l)}{\prod_{k=1}^{m}(z-x_{i_k})}. \tag{3.16}$$

and construct, using the polynomial $R_{i_1,\ldots,i_m}(z)$, the rational function

$$S_j^{i_1,\ldots,i_m}(z) = \frac{D_j(R'_{i_1,\ldots,i_m}(z)) - D_{j+1}(R_{i_1,\ldots,i_m}(z))}{R'_{i_1,\ldots,i_m}(z)}, \tag{3.17}$$

where D_j is the umbral derivative of order j.

Proposition 3.3. *Let $[\varepsilon]$ be the fundamental characteristic with the lower bounds fixed as in* (3.14)*. Then*

$$\mathfrak{R}_j^n = \frac{1}{2}\sum_{k=1}^{g} y_k \left(\sum_{i\in\boldsymbol{I}} S_j^{\boldsymbol{I}\setminus\{i\}}(x_k) - (n-1)S_j^{\boldsymbol{I}}(x_k)\right) + \frac{1}{2}\sum_{i\in\boldsymbol{I}} y_i S_j^{\boldsymbol{I}\setminus\{i\}}(x_i). \tag{3.18}$$

The function $\mathfrak{R}_j^n(x, y; \xi, \eta)$ can be given in various forms. In particular, for the case $N = 1$, we have

Proposition 3.4. *The following formula is valid*

$$\mathfrak{R}_j^1(z, w; \boldsymbol{u}) = \frac{1}{2} Z_j(z, w; \boldsymbol{u}) - \frac{1}{2}\mathfrak{Z}_j(\boldsymbol{u}).$$

In this formula $Z_j(z, w; \boldsymbol{u})$ is a special logarithmic derivative of the polynomial $\mathcal{P}(z; \boldsymbol{u})$,

$$Z_j(z, w; \boldsymbol{u}) = \frac{(wD_j + \partial_j)\mathcal{P}(z; \boldsymbol{u})}{2\mathcal{P}(z; \boldsymbol{u})}, \tag{3.19}$$

where D_j is the umbral derivative of order j and ∂_j is the standard derivative with respect to the variable u_j. The function $\mathfrak{Z}_j(\boldsymbol{u})$ can be given as [1]

$$\mathfrak{Z}_j(\boldsymbol{u}) = -\frac{1}{2}\sum_{k,l=1}^{g} x_k^l \frac{\partial x_k}{\partial u_{j+l}}, \tag{3.20}$$

where $\{P_1, \ldots, P_g\}$ is the Abelian preimage of the point $\boldsymbol{u} \in \mathrm{Jac}(V)$.

In particular, the following expressions are valid for $\mathfrak{Z}_j(\boldsymbol{u})$

$$\begin{aligned} \mathfrak{Z}_g(\boldsymbol{u}) &= 0, \\ \mathfrak{Z}_{g-1}(\boldsymbol{u}) &= \mathfrak{P}_{gg}(\boldsymbol{u}), \\ \mathfrak{Z}_{g-2}(\boldsymbol{u}) &= \mathfrak{P}_g(\boldsymbol{u})\mathfrak{P}_{gg}(\boldsymbol{u}) + 2\mathfrak{P}_{g-1,g}(\boldsymbol{u}). \end{aligned} \tag{3.21}$$

The formulas given above represent the addition theorem of the kind "point+divisor" for the Kleinian ζ-function. In the case $g = 1$ the formula (3.11) represents the addition theorem for the Weierstrass ζ-function,

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\left[\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right].$$

[1]We note that the expression for the functions $\mathfrak{Z}_j(\boldsymbol{u})$ given in the monograph [2] on page 321 is correct only at the values $j = g$ and $j = g-1$ and is *wrong* for $j < g-1$.

on the elliptic curve $y^2 = f(x) = 4x^3 - g_2x - g_3$.

3.4. **Solution of the Jacobi inversion problem.** The ζ-formula allows us to express symmetrical functions of the divisor $P_1, \dots, P_g$ in terms of the hyperelliptic Kleinian functions and to solve the Jacobi inversion problem.

Introduce the elementary symmetric functions e_r with the help of the generating function

$$E(t) = \sum_{r\geq 0} e_r t^r = \prod_{i\geq 0}(1 + x_i t). \tag{3.22}$$

Thus

$$e_1 = \sum_{k=1}^{g} x_k, \quad e_2 = \sum_{1\leq k<i\leq n} x_i x_k, \dots, e_g = \prod_{k=1}^{g} x_k.$$

Replacing x_i by $1/x_i$ we find

$$\widetilde{E}(t) = \sum_{r\geq 0} \tilde{e}_r t^r = \prod_{i\geq 0}\left(1 + \frac{t}{x_i}\right). \tag{3.23}$$

In this case

$$\tilde{e}_1 = \sum_{k=1}^{g} \frac{1}{x_k}, \quad \tilde{e}_2 = \sum_{1\leq k<i\leq n} \frac{1}{x_i x_k}, \dots, \tilde{e}_g = \prod_{k=1}^{g} \frac{1}{x_k}.$$

Theorem 3.5. *Let V be the even curve of genus g, $\lambda_{2g+2} \neq 0$. Then the Abel preimage of the point $\boldsymbol{u} \in \mathrm{Jac}(V)$ is given by the set $\{P_1, \dots, P_g\} \in \mathrm{symm}^g V$, where $\{x_1, \dots, x_g\}$ are the zeros of the polynomial*

$$\mathcal{P}(x; \boldsymbol{u}) = x^g - x^{g-1}\mathfrak{P}_g(\boldsymbol{u}) - x^{n-2}\mathfrak{P}_{g-1}(\boldsymbol{u}) - \dots - \mathfrak{P}_1(\boldsymbol{u}), \tag{3.24}$$

where

$$\mathfrak{P}_i(\boldsymbol{u}) = \frac{1}{\sqrt{\lambda_{2g+2}}}\left\{\zeta_i\left(\boldsymbol{u} + \int_{Q_0}^{P_{\infty,+}} \mathrm{d}\mathbf{u}\right) - \zeta_i\left(\boldsymbol{u} + \int_{Q_0}^{P_{\infty,-}} \mathrm{d}\mathbf{u}\right)\right\} + c_j, \tag{3.25}$$

with the base point of the Abel map being chosen as a branch point. The constants c_j are principal parts of the poles of order $g - j + 1$ in the expansion

$$\frac{1}{\sqrt{\lambda_{2g+2}}}\sqrt{\sum_{k=0}^{2g+2}\frac{\lambda_k}{\xi^k}} = \frac{1}{\xi^{g+1}} + \frac{c_1}{\xi^g} + \frac{c_2}{\xi^{g-1}} + \dots + \frac{c_g}{\xi}.$$

In particular,

$$\begin{aligned} c_g &= \frac{1}{2}\frac{\lambda_{2g+1}}{\lambda_{2g+2}}, \\ c_{g-1} &= \frac{1}{2}\frac{\lambda_{2g}}{\lambda_{2g+2}} - \frac{1}{8}\frac{\lambda_{2g+1}^2}{\lambda_{2g+2}^2}, \\ c_{n-2} &= \frac{1}{2}\frac{\lambda_{2g-1}}{\lambda_{2g+2}} - \frac{1}{4}\frac{\lambda_{2g+1}\lambda_{2g}}{\lambda_{2g+2}^2} + \frac{1}{16}\frac{\lambda_{2g+1}^3}{\lambda_{2g+2}^3}, \quad \text{etc.} \end{aligned}$$

The coordinates $\{y_1, \dots, y_g\}$ are then given by

$$y_k = -\frac{\partial \mathcal{P}(x; \boldsymbol{u})}{\partial u_g}\Big|_{x=x_k}. \tag{3.26}$$

Proof. Compute

$$\lim_{z\to\infty^+} \zeta_i\left(\boldsymbol{u} + \int_{Q_0}^{(z,w)} \mathrm{d}\mathbf{u}\right) - \lim_{z\to\infty^-} \zeta_i\left(\boldsymbol{u} + \int_{Q_0}^{(z,w)} \mathrm{d}\mathbf{u}\right)$$

with the aid of the ζ-formula (3.11).

We find from the equations of the Abel map that

$$\sum_{i=1}^{g} \frac{x_i^{k-1}}{y_i}\frac{\partial x_i}{\partial u_j} = \delta_{jk}, \qquad \frac{\partial x_k}{\partial u_g} = \frac{y_k}{\prod_{i\neq k}(x_k - x_i)}.$$

On the other hand we have

$$\frac{\partial \mathcal{P}}{\partial u_g}\Big|_{x=x_k} = -\frac{\partial x_k}{\partial u_g}\prod_{i\neq k}(x_i - x_k),$$

and we obtain (3.26). □

Analogously one can express other differences of ζ-functions,

$$\tilde{e}_{g-j+1} = \frac{(-1)^{j+1}}{\sqrt{\lambda_0}}\left\{\zeta_j\left(\boldsymbol{u} + \int_{Q_0}^{P_{0,+}} \mathrm{d}\mathbf{u}\right) - \zeta_j\left(\boldsymbol{u} - \int_{Q_0}^{P_{0,+}} \mathrm{d}\mathbf{u}\right)\right\} - 2\int_{Q_0}^{P_{0,+}} \mathrm{d}r_{g-j+1}$$

and also

$$\begin{aligned}\zeta_{g-j+1}\left(\boldsymbol{u} + \int_{Q_0}^{P_{0,+}} \mathrm{d}\mathbf{u}\right) &- \zeta_{g-j+1}\left(\boldsymbol{u} + \int_{Q_)}^{P_{\infty,+}} \mathrm{d}\mathbf{u}\right)\\ &= -2\int_{Q_0}^{P_{\infty,+}} \mathrm{d}r_{g-j+1} + \frac{(-1)^j}{2}\left(e_i\sqrt{\lambda_{2g+2}} + (-1)^g\tilde{e}_{n-i+1}\sqrt{\lambda_0}\right)\\ &\qquad + \frac{(-1)^j}{2}\sum_{i=1}^{g}\frac{y_i e_{j-1}^{(i)}}{x_i\mathcal{P}'(x_i)} + \frac{1}{2}c_{g-j+1},\end{aligned}$$

where $\mathcal{P} = \prod_{k=1}^{g}(x - x_i)$ and $e_k^{(l)}$ are elementary symmetric functions of order k of $g-1$ elements $\{x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_g\}$.

Some these formulas were derived by Abenda and Fedorov [1].

3.5. **Special relations for ζ-functions.** Let

$$\boldsymbol{\Delta}_0 = \int_{Q_0}^{P_{0,+}} \mathrm{d}\mathbf{u}, \quad \boldsymbol{\Delta}_\infty = \int_{Q_0}^{P_{\infty,+}} \mathrm{d}\mathbf{u}, \quad \boldsymbol{\Delta} = \int_{P_{0,+}}^{P_{\infty,+}} \mathrm{d}\mathbf{u} = \boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty, \tag{3.27}$$

where $P_{0_\pm} = (0, \pm\sqrt{R(0)}) = (0, y_{0_\pm})$, and introduce

$$\partial_{\boldsymbol{W}_\infty} f(u) = \sum_{i=1}^{\infty} W_{\infty,j}\frac{\partial f}{\partial u_i}, \quad \partial_{\boldsymbol{W}_0} f(u) = \sum_{i=1}^{\infty} W_{0,j}\frac{\partial f}{\partial u_i}, \tag{3.28}$$

where $W_{\infty,i}$ and $W_{0,i}$ are the coefficients of in the expansion of the ith normalized holomorphic differential $\mathrm{d}v_i$ the vicinity of ∞ and 0, respectively. We shall derive the following relations.

Lemma 3.6. *The following relations are valid*

$$\begin{aligned}&\frac{\theta[\upsilon](2\boldsymbol{\Delta}_0|\tau)\partial_{\boldsymbol{W}_\infty}\theta[\upsilon](0|\tau)}{\theta[\upsilon](\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_\infty|\tau)\theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau)}\,\frac{\theta(\boldsymbol{u} + \boldsymbol{\Delta}_\infty|\tau)\,\theta(\boldsymbol{u} - \boldsymbol{\Delta}_\infty|\tau)}{\theta(\boldsymbol{u} + \boldsymbol{\Delta}_0|\tau)\,\theta(\boldsymbol{u} - \boldsymbol{\Delta}_0|\tau)}\\ &\qquad = \partial_{\boldsymbol{W}_\infty}\ \ln\left\{\frac{\theta[\upsilon](\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_\infty|\tau)}{\theta[\upsilon](\boldsymbol{\Delta}_\infty - \boldsymbol{\Delta}_0|\tau)}\frac{\theta(\boldsymbol{u} - \boldsymbol{\Delta}_0|\tau)}{\theta(\boldsymbol{u} + \boldsymbol{\Delta}_0|\tau)}\right\}\end{aligned} \tag{3.29}$$

and

$$\frac{\theta[\upsilon](\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_\infty|\tau)\partial_{\boldsymbol{W}_\infty}\theta[\upsilon](0|\tau)}{\theta[\upsilon](2\boldsymbol{\Delta}_\infty|\tau)\theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau)}\,\frac{\theta(\boldsymbol{u} + \boldsymbol{\Delta}_\infty|\tau)\,\theta(\boldsymbol{u} - 2\boldsymbol{\Delta}_\infty + \boldsymbol{\Delta}_0|\tau)}{\theta(\boldsymbol{u} + \boldsymbol{\Delta}_0|\tau)\,\theta(\boldsymbol{u} - \boldsymbol{\Delta}_\infty|\tau)}$$

$$= \partial_{\boldsymbol{W}_\infty} \ln \left\{ \frac{\theta[v]\,(\boldsymbol{\Delta}_\infty + \boldsymbol{\Delta}_0|\tau)}{\theta[v]\,(2\boldsymbol{\Delta}_\infty|\tau)} \frac{\theta\,(\boldsymbol{u} - \boldsymbol{\Delta}_\infty|\tau)}{\theta\,(\boldsymbol{u} + \boldsymbol{\Delta}_0|\tau)} \right\}. \quad (3.30)$$

Proof. Represent the argument $\boldsymbol{u}$ as the Abelian image of the nonspecial divisor $P_1 + \ldots + P_g - (0, E_1) - \ldots - (0, E_g)$ and consider the right- and left-hand sides of the equalities (3.29) and (3.30) as functions of variable $P_1 \in V$. To prove (3.29) and (3.30) we must first show that the right- and left-hand sides satisfy the following conditions: (**i**) they have the same periodicity property when the variable P_1 goes round each $\mathfrak{a}_i$ and $\mathfrak{b}_i$-cycle, $i = 1, \ldots, g$; (**ii**) they have the same poles; (**iii**) they have the same residues at the poles.

Property (**i**) is proved by the fact that in the both cases the right- and left-hand sides are Abelian function of the variable $\boldsymbol{u}$, which follows from the transformation properties in shift of periods of the θ-function (2.11).

The poles of the right- and left-hand sides of the equality (3.29), according to the Riemann theorem on zeros of θ-function, are the following

$$2\bar{P}_2, \ldots, 2\bar{P}_g, P_{0,+}, P_{0,-},$$

where $\bar{P}$ denotes the point conjugated with $P = (y, x)$: $\bar{P} = (-y, x)$.

Analogously in the case of (3.30) they are

$$2\bar{P}_2, \ldots, 2\bar{P}_g, P_{0,-}, P_{\infty,+}.$$

Property (**iii**) is proved by considering the constants in front of the $\boldsymbol{u}$-dependent part chosen.

When conditions (**i**)–(**iii**) are satisfied that the right hand side and left hand sides of (3.30) and (3.30) can differ by a constants. The last are computed by substituting the special values of $\boldsymbol{u}$.

□

We shall use these formulas below in the analysis of the Thirring model.

4. θ-function solutions of the Thirring model

In this Section we formulate the θ-function solution of the Thirring equation (1.1) by following [27]. We then show that the solution is valid because of ζ-relations derived in the preceeding section.

Let Υ be an odd non-singular half-period. Then

$$\theta(\boldsymbol{z} + \Upsilon) = -\exp\{-2i\pi \boldsymbol{v}^T(\boldsymbol{z} + \frac{1}{2}\boldsymbol{v}\tau)\}\theta[v](\boldsymbol{z}), \quad (4.1)$$

where $[v]$ is the characteristic of the vector Υ.

Assume the curve V to be nonsingular (i.e., $E_m \neq E_{m'}$ for $m \neq m'$, $m, m' = 0, \ldots, 2g+1$) and $g \in \mathbb{N}$. Define a normal differential of the third kind, with simple poles at $P_{0,-}$ and $P_{\infty,-}$ with residues $+1$ and -1, vanishing $\mathfrak{a}$-periods, and being otherwise holomorphic on V. It can be written as

$$\mathrm{d}\omega_{P_{0,-}P_{\infty,-}} = \frac{y + y_{0,-}}{2z}\frac{dz}{y} + \frac{\prod_{j=1}^{g}(z - \Lambda_j)dz}{2y}, \quad P_{0,-} = (0, y_{0,-}), \quad (4.2)$$

where $\{\Lambda_j\}_{j=1,\ldots,n}$ are uniquely determined by the normalization

$$\int_{\mathfrak{a}_j} \mathrm{d}\omega_{P_{0,-},P_{\infty,-}} = 0, \quad j = 1, \ldots, g. \quad (4.3)$$

The explicit formula (4.2) then implies (using the local coordinate $\xi = z$ near $P_{0,\pm}$)

$$\mathrm{d}\omega_{P_{0,-},P_{\infty,-}}(P) \underset{\xi \to 0}{=} \begin{Bmatrix} \xi^{-1} \\ 0 \end{Bmatrix} d\xi \pm \left(\sum_{q=0}^{\infty} (q+1)\omega^0_{q+1}\xi^q \right) d\xi \text{ as } P \to P_{0,\mp}, \quad (4.4)$$

and similarly (using the local coordinate $\xi = 1/z$ near $P_{\infty,-}$),

$$\mathrm{d}\omega_{P_{0,-},P_{\infty,-}}(P) \underset{\xi\to 0}{=} \begin{Bmatrix} -\xi^{-1} \\ 0 \end{Bmatrix} d\xi \pm \left(\sum_{q=0}^{\infty}(q+1)\omega_{q+1}^{\infty}\xi^q\right) d\xi \text{ as } P \to P_{0,\mp}. \tag{4.5}$$

In particular,

$$\int_{Q_0}^{P} \mathrm{d}\omega_{P_{0,-},P_{\infty,-}} \underset{\xi\to 0}{=} \begin{Bmatrix} \ln(\xi) \\ 0 \end{Bmatrix} + \omega_0^{0,\mp} \pm \omega_1^0\xi \pm \omega_2^0\xi^2 + O(\xi^3) \text{ as } P \to P_{0,-}, \tag{4.6}$$

$$\int_{Q_0}^{P} \mathrm{d}\omega_{P_{0,-},P_{\infty,0}} \underset{\xi\to 0}{=} \begin{Bmatrix} -\ln(\xi) \\ 0 \end{Bmatrix} + \omega_0^{\infty\mp} \pm \omega_1^{\infty}\xi \pm \omega_2^{\infty}\xi^2 + O(\xi^3) \text{ as } P \to P_{\infty,\mp}. \tag{4.7}$$

Here Q_0 is an appropriate base point of the Abel map and we use the same path of integration from Q_0 to P in all Abelian integrals in this section.

A comparison of (4.4), (4.5) with (4.2) then yields

$$\omega_1^0 = \frac{1}{4}\sum_{m=0}^{2g+1}\frac{1}{E_m} - \frac{(-1)^g}{2y_{0,+}}\prod_{j=1}^{g}\Lambda_j, \tag{4.8}$$

$$\omega_1^{\infty} = -\frac{1}{4}\sum_{m=0}^{2g+1}E_m + \frac{1}{2}\sum_{j=1}^{g}\Lambda_j. \tag{4.9}$$

Next, we intend to go a step further and derive alternative expressions for the expansion coefficients $\omega_0^{0,\pm}$, ω_1^0, $\omega_0^{\infty\pm}$, and ω_1^{∞} in (4.6) and (4.7).

Lemma 4.1. *Let $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Delta}_\infty$ be as in* (3.27) *and $\partial_{\boldsymbol{W}_\infty}$ and $\partial_{\boldsymbol{W}_0}$ as in* (3.28). *Let $[v]$ be a nonsingular odd characteristic. Then the expansion of the normalized third kind integral $\int_{Q_0}^{P}\mathrm{d}\omega_{P_{0,-},P_{\infty_-}}$ in the vicinity of the points $P_{0_\pm}$ and $P_{\infty_\pm}$ is given, up to a common additive constant, as follows*

$$\begin{aligned}\int_{Q_0}^{P}\omega_{P_{0,-},P_{\infty_-}}(P) &= \ln\frac{\theta[v]\,(2\boldsymbol{\Delta}_0|\tau)}{\theta[v]\,(\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty|\tau)} + \frac{1}{y_{0,+}}\partial_{\boldsymbol{W}_0}\ln\frac{\theta[v]\,(2\boldsymbol{\Delta}_0|\tau)}{\theta[v]\,(\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty|\tau)}\xi + O(\xi)\\ &\qquad \textit{as } P\to P_{0,+},\ z=\xi,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\int_{Q_0}^{P}\omega_{P_{0,-},P_{\infty_-}}(P) &= \ln\xi + \ln\left\{\frac{\partial_{\boldsymbol{W}_0}\theta[v](0)}{y_{0,+}\theta[v](\boldsymbol{\Delta_0}-\boldsymbol{\Delta_\infty}|\tau)}\right\}\\ &\quad + \left(\frac{1}{4}\sum_{k=0}^{2g+1}\frac{1}{E_k} - \frac{1}{y_{0,+}}\partial_{\boldsymbol{W}_0}\ln\ \theta[v](\boldsymbol{\Delta}_0-\boldsymbol{\Delta}_\infty|\tau\right)\xi + O(\xi^2)\\ &\qquad \textit{as } P\to P_{0,-},\ z=\xi,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\int_{Q_0}^{P}\omega_{P_{0,-},P_{\infty_-}}(P) &= \frac{\theta[v]\,(\boldsymbol{\Delta}_\infty+\boldsymbol{\Delta}_0|\tau)}{\theta[v]\,(2\boldsymbol{\Delta}_\infty)} - \partial_{\boldsymbol{W}_\infty}\ln\frac{\theta[v](\boldsymbol{\Delta}_\infty+\boldsymbol{\Delta}_0|\tau)}{\theta[v](2\boldsymbol{\Delta}_\infty|\tau)}\xi + O(\xi^2),\\ &\qquad \textit{as } P\to P_{\infty,+},\ z=1/\xi\end{aligned} \tag{4.12}$$

$$\int_{Q_0}^{P}\omega_{P_{0,-},P_{\infty_-}}(P) = -\ln\xi - \ln\frac{\partial_{\boldsymbol{W}_\infty}\theta[v](0|\tau)}{\theta[v](\boldsymbol{\Delta_0}-\boldsymbol{\Delta_\infty}|\tau)}$$

$$-\left(\frac{1}{4}\sum_{k=0}^{2g+1} E_k - \partial_{\boldsymbol{W}_\infty} \ln\, \theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau)\right)\xi + O(\xi^2),$$

$$\text{as } P \to P_{\infty,-},\ z = 1/\xi. \tag{4.13}$$

Proof. The differential (4.2) is uniquely defined by its poles and the normalization condition (4.3). Equivalently this differential can be given in the form

$$\mathrm{d}\omega_{P_{0,-},P_{\infty,-}}(P) = \mathrm{d}\, \ln \frac{\theta[\upsilon]\left(\int_{Q_0}^{P} \mathrm{d}\mathbf{u} - \int_{Q_0}^{P_{0,-}} \mathrm{d}\mathbf{u}|\tau\right)}{\theta[\upsilon]\left(\int_{Q_0}^{P} \mathrm{d}\mathbf{u} - \int_{Q_0}^{P_{\infty,-}} \mathrm{d}\mathbf{u}|\tau\right)}.$$

The integration of this differential with the previously described expansion at the corresponding points give the above expressions (4.10)–(4.13). □

We remark that the properties $\omega_k^{0\pm} = -\omega_k^{0\mp}$ and $\omega_k^{\infty\pm} = -\omega_k^{\infty\mp}$, $k = 1, 2, \ldots$, which follows from the explicit realization of the differential of the third kind in the form (4.2), lead to a set of identities (special addition theorems), the first pair (for $k = 1$) being

$$\partial_{W_0} \ln \frac{\theta[\upsilon](2\boldsymbol{\Delta}_0|\tau)}{\theta[\upsilon](\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_\infty|\tau)\theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau)} = -\frac{1}{4} y_{0,+} \sum_{k=0}^{2g+1} \frac{1}{E_k}, \tag{4.14}$$

$$\partial_{W_\infty} \ln \frac{\theta[\upsilon](2\boldsymbol{\Delta}_\infty|\tau)}{\theta[\upsilon](\boldsymbol{\Delta}_\infty + \boldsymbol{\Delta}_0|\tau)\theta[\upsilon](\boldsymbol{\Delta}_\infty - \boldsymbol{\Delta}_0|\tau)} = \frac{1}{4} \sum_{k=0}^{2g+1} E_k. \tag{4.15}$$

Taking into account these equivalences as well as the expansions (4.10)–(4.13), we fix the following expressions for the coefficients of the expansion of the differential of the third kind

$$\omega_0^{0,+} = \ln \frac{\theta[\upsilon]\,(2\boldsymbol{\Delta}_0|\tau)\,\theta[\upsilon]\,(\boldsymbol{\Delta}_\infty|\tau)}{\theta[\upsilon]\,(\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_\infty|\tau)\,\theta[\upsilon](\boldsymbol{\Delta}_0)}, \tag{4.16}$$

$$\omega_0^{0,-} = \ln \frac{\partial_{\boldsymbol{W}_0}\theta[\upsilon](0|\tau)\theta[\upsilon]\,(\boldsymbol{\Delta}_\infty|\tau)}{g_{g+2}\theta[\upsilon](\boldsymbol{\Delta_0} - \boldsymbol{\Delta_\infty}|\tau)\theta[\upsilon]\,(\boldsymbol{\Delta}_0|\tau)}, \tag{4.17}$$

$$\omega_0^{\infty,+} = \ln \frac{\theta[\upsilon]\,(\boldsymbol{\Delta}_\infty + \boldsymbol{\Delta}_0|\tau)\,\theta[\upsilon]\,(\boldsymbol{\Delta}_\infty|\tau)}{\theta[\upsilon]\,(2\boldsymbol{\Delta}_\infty)\,\theta[\upsilon]\,(\boldsymbol{\Delta}_0|\tau)}, \tag{4.18}$$

$$\omega_0^{\infty,-} = \ln \frac{\theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau)\theta[\upsilon]\,(\boldsymbol{\Delta}_\infty|\tau)}{\partial_{\boldsymbol{W}_\infty}\theta[\upsilon](0|\tau)\theta[\upsilon]\,(\boldsymbol{\Delta}_0|\tau)}, \tag{4.19}$$

$$\omega_1^0 = -\frac{1}{4}\sum_{k=0}^{2g+1} \frac{1}{E_k} + \frac{1}{y_{0,+}} \partial_{\boldsymbol{W}_0} \ln\, \theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau), \tag{4.20}$$

$$\omega_1^\infty = \frac{1}{4}\sum_{k=0}^{2g+1} E_k - \partial_{\boldsymbol{W}_\infty} \ln\, \theta[\upsilon](\boldsymbol{\Delta}_0 - \boldsymbol{\Delta}_\infty|\tau), \tag{4.21}$$

where $\omega_1^0 = \omega_1^{0,+} = -\omega_1^{0,-}$ and $\omega_1^\infty = \omega_1^{\infty,+} = -\omega_1^{\infty,-}$.

The θ-function solutions of the Thirring model are given as follows. Let $\boldsymbol{\Delta}_0$ and $\boldsymbol{\Delta}_\infty$ be as in (3.27) and $\partial_{\boldsymbol{W}_\infty}$ and $\partial_{\boldsymbol{W}_0}$ as in (3.28). Denote the linear winding vector as

$$\boldsymbol{\mathfrak{L}}(x,t) = 2i\left(\boldsymbol{W}_\infty x + \boldsymbol{W}_0 t\right).$$

The θ-function solution of the Thirring model derived in [27] has the form

$$u(x,t) = -C_0^{-1} e^{-\omega_0^{0,+}} \frac{\theta\left(\boldsymbol{\mathfrak{L}}(x,t) - \boldsymbol{\Delta_0}|\tau\right)}{\theta\left(\boldsymbol{\mathfrak{L}}(x,t) - \boldsymbol{\Delta}_\infty|\tau\right)} e^{-2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.22}$$

$$u^*(x,t) = C_0^{-1} e^{\omega_0^{0,-}} \frac{\theta\left(\boldsymbol{\mathfrak{L}}(x,t) + 2\boldsymbol{\Delta_0} - \boldsymbol{\Delta_\infty}|\tau\right)}{\theta\left(\boldsymbol{\mathfrak{L}}(x,t) + \boldsymbol{\Delta}_0|\tau\right)} e^{2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.23}$$

$$v(x,t) = -C_0^{-1} e^{-\omega_0^{\infty-}} \frac{\theta\left(\boldsymbol{\mathcal{L}}(x,t)+\boldsymbol{\Delta}_\infty|\tau\right)}{\theta\left(\boldsymbol{\mathcal{L}}(x,t)+\boldsymbol{\Delta}_0|\tau\right)} e^{-2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.24}$$

$$v^*(x,t) = C_0^{-1} e^{\omega_0^{\infty+}} \frac{\theta\left(\boldsymbol{\mathcal{L}}(x,t)-2\boldsymbol{\Delta}_\infty+\boldsymbol{\Delta}_0|\tau\right)}{\theta\left(\boldsymbol{\mathcal{L}}(x,t)-\boldsymbol{\Delta}_\infty|\tau\right)} e^{2i(\omega_1^\infty x - \omega_1^0 t)}. \tag{4.25}$$

Direct substitution of the θ-function formulas (4.22)–(4.25) into the first Thirring equation yields

$$\begin{aligned}
&-\frac{1}{4}\sum_{k=0}^{2g+1} E_k - \partial_{\boldsymbol{W}_\infty} \ln \left\{ \frac{\theta[\upsilon](2\boldsymbol{\Delta}_\infty|\tau)}{\theta[\upsilon](\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty|\tau)} \frac{\theta\left(\boldsymbol{\mathcal{L}}(x,t)-\boldsymbol{\Delta}_\infty|\tau\right)}{\theta\left(\boldsymbol{\mathcal{L}}(x,t)-\boldsymbol{\Delta}_0|\tau\right)} \right\} \\
&+ \frac{\theta[\upsilon](2\boldsymbol{\Delta}_0|\tau)\partial_{\boldsymbol{W}_\infty}\theta[\upsilon](0|\tau)}{\theta[\upsilon](\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty|\tau)\theta[\upsilon](\boldsymbol{\Delta}_0-\boldsymbol{\Delta}_\infty|\tau)} \frac{\theta\left(\boldsymbol{\mathcal{L}}(x,t)+\boldsymbol{\Delta}_\infty|\tau\right)\theta\left(\boldsymbol{\mathcal{L}}(x,t)-\boldsymbol{\Delta}_\infty|\tau\right)}{\theta\left(\boldsymbol{\mathcal{L}}(x,t)+\boldsymbol{\Delta}_0|\tau\right)\theta\left(\boldsymbol{\mathcal{L}}(x,t)-\boldsymbol{\Delta}_0|\tau\right)} \\
&- \frac{\theta[\upsilon](\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty)\partial_{\boldsymbol{W}_\infty}\theta[\upsilon](0|\tau)}{\theta[\upsilon](2\boldsymbol{\Delta}_\infty|\tau)\theta[\upsilon](\boldsymbol{\Delta}_0-\boldsymbol{\Delta}_\infty)} \frac{\theta\left(\boldsymbol{\mathcal{L}}(x,t)+\boldsymbol{\Delta}_\infty|\tau\right)\theta\left(\boldsymbol{\mathcal{L}}(x,t)-2\boldsymbol{\Delta}_\infty+\boldsymbol{\Delta}_0|\tau\right)}{\theta\left(\boldsymbol{\mathcal{L}}(x,t)+\boldsymbol{\Delta}_0|\tau\right)\theta\left(\boldsymbol{\mathcal{L}}(x,t)-\boldsymbol{\Delta}_\infty|\tau\right)} = 0.
\end{aligned} \tag{4.26}$$

We use the formulas (3.29) and (3.29) of Lemma 3.6. The right-hand side of the equality has three terms. The first one can be written in the form

$$\begin{aligned}
&-\partial_{\boldsymbol{W}_\infty} \ln \theta[\upsilon](\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty|\tau) - \partial_{\boldsymbol{W}_\infty} \ln \theta[\upsilon](2\boldsymbol{\Delta}_\infty|\tau) \\
&\qquad - \partial_{\boldsymbol{W}_\infty} \ln \theta(\boldsymbol{\mathcal{L}}-\boldsymbol{\Delta}_0|\tau) + \partial_{\boldsymbol{W}_\infty} \ln \theta(\boldsymbol{\mathcal{L}}-\boldsymbol{\Delta}_\infty|\tau)
\end{aligned}$$

whilst the remaining two are

$$\begin{aligned}
&\partial_{\boldsymbol{W}_\infty} \ln \theta(\boldsymbol{\mathcal{L}}-\boldsymbol{\Delta}_0|\tau) + \partial_{\boldsymbol{W}_\infty} \ln \theta(\boldsymbol{\mathcal{L}}+\boldsymbol{\Delta}_0|\tau) \\
&\qquad + \partial_{\boldsymbol{W}_\infty} \ln \theta[\upsilon](\boldsymbol{\Delta}_0+\boldsymbol{\Delta}_\infty|\tau) - \partial_{\boldsymbol{W}_\infty} \ln \theta[\upsilon](\boldsymbol{\Delta}_\infty-\boldsymbol{\Delta}_0|\tau)
\end{aligned}$$

and

$$\begin{aligned}
&-\partial_{\boldsymbol{W}_\infty} \ln \theta(\boldsymbol{\mathcal{L}}-\boldsymbol{\Delta}_\infty|\tau) + \partial_{\boldsymbol{W}_\infty} \ln \theta(\boldsymbol{\mathcal{L}}+\boldsymbol{\Delta}_0|\tau) \\
&\qquad + \partial_{\boldsymbol{W}_\infty} \ln \theta[\upsilon](\boldsymbol{\Delta}_\infty-\boldsymbol{\Delta}_0|\tau) - \partial_{\boldsymbol{W}_\infty} \ln \theta[\upsilon](2\boldsymbol{\Delta}_\infty|\tau),
\end{aligned}$$

respectively. Their sum vanishes because of the equality (4.15).

The substitution of the θ-function formulas (4.22)–(4.25) in the second equation in the Thirring equations is reduced in the same way to the equivalence (4.14).

5. Elliptic and soliton solutions of the Thirring model

The soliton solution of the Thirring equations (1.1) was derived in [37] in the framework of the inverse scattering method. In our notation this solution reads

$$u(x,t) = \frac{\sin\phi}{\sqrt{r}} \frac{\exp\left\{2i(rx+\frac{1}{r}t+\theta_0)\right\}}{\cosh\left(2\sin\phi\,(r(x-x_0)-\frac{1}{r}t)+i\frac{\phi}{2}\right)}, \tag{5.1}$$

$$v(x,t) = -\sin\phi\sqrt{r} \frac{\exp\left\{2i(rx+\frac{1}{r}t+\theta_0)\right\}}{\cosh\left(2\sin\phi\,(r(x-x_0)-\frac{1}{r}t)-i\frac{\phi}{2}\right)}, \tag{5.2}$$

where x_0, θ_0 are arbitrary and r is a parameter; the validity of these formulas can be also checked by direct substitution into the equations (1.1)

Elliptic solutions of the Thirring model were discussed by Holod and Prikarpatski [33] and also Kamchatnov, Steudel, and Zabolotski [34][2]. In both papers the elliptic solutions were obtained as the result of straightforward inversions of the elliptic integrals but the general θ-function expressions for the fields $u(x,t)$, $v(x,t)$ (see below (5.4), (5.6) and (5.29), (5.30) for a special case) were not clarified. Here we shall derive elliptic solutions by specializing the general formulas (4.22)–(4.25) to

[2]The authors are grateful to N. Kostov for pointing out this paper.

the case of an elliptic curve and obtain the Kuznetsov–Mikhailov soliton solution as a limiting case of the elliptic function.

5.1. **Elliptic solutions.** Consider the even elliptic curve[3]

$$\mathcal{K}_1\colon y^2 = (z-E_0)(z-E_1)(z-E_2)(z-E_3), \tag{5.3}$$

where $E_0, \dots, E_3$ are arbitrary complex numbers. Let us restrict the formulas (4.22)–(4.25) to the curve (5.3). The vector of Riemann constants in the case where the half period is $\frac{1}{2}+\frac{1}{2}\tau$ has the characteristic

$$\left[\begin{array}{c}\frac{1}{2}\\ \frac{1}{2}\end{array}\right].$$

The elliptic solution then has the form

$$u(x,t) = -C_0^{-1} e^{-\omega_0^{0,+}} \frac{\vartheta_1\big(\mathcal{L}(x,t)-\Delta_0|\tau\big)}{\vartheta_1\big(\mathcal{L}(x,t)-\Delta_\infty|\tau\big)} e^{i\pi(\Delta_0-\Delta_\infty)} e^{-2i(\omega_1^\infty x-\omega_1^0 t)}, \tag{5.4}$$

$$u^*(x,t) = C_0^{-1} e^{\omega_0^{0,+}} \frac{\vartheta_1\big(\mathcal{L}(x,t)+2\Delta_0-\Delta_\infty|\tau\big)}{\vartheta_1\big(\mathcal{L}(x,t)-\Delta_0|\tau\big)} e^{-i\pi(\Delta_0-\Delta_\infty)} e^{2i(\omega_1^\infty x-\omega_1^0 t)}, \tag{5.5}$$

$$v(x,t) = -C_0^{-1} e^{-\omega_0^{\infty,-}} \frac{\vartheta_1\big(\mathcal{L}(x,t)+\Delta_\infty|\tau\big)}{\vartheta_1\big(\mathcal{L}(x,t)+\Delta_0|\tau\big)} e^{i\pi(\Delta_0-\Delta_\infty)} e^{-2i(\omega_1^\infty x-\omega_1^0 t)}, \tag{5.6}$$

$$v^*(x,t) = C_0^{-1} e^{\omega_0^{\infty,-}} \frac{\vartheta_1\big(\mathcal{L}(x,t)-2\Delta_\infty+\Delta_0|\tau\big)}{\vartheta_1\big(\mathcal{L}(x,t)-\Delta_\infty|\tau\big)} e^{-i\pi(\Delta_0-\Delta_\infty)} e^{2i(\omega_1^\infty x-\omega_1^0 t)}. \tag{5.7}$$

In these formulas C_0 is a constant which we shall fix later. The parameters $\omega_0^{0,+}$, $\omega_0^{\infty,-}$, ω_1^0, ω_1^∞ are given by the formulas

$$\omega_0^{0,+} = \ln \frac{\vartheta_1(2\Delta_0|\tau)\vartheta_1(\Delta_\infty|\tau)}{\vartheta_1(\Delta_0+\Delta_\infty|\tau)\vartheta_1(\Delta_0|\tau)}, \tag{5.8}$$

$$\omega_0^{\infty,-} = \ln \frac{\vartheta_1(\Delta_0-\Delta_\infty|\tau)\vartheta_1(\Delta_\infty|\tau)}{W\vartheta_1'(0|\tau)\vartheta_1(\Delta_0|\tau)}, \tag{5.9}$$

$$\omega_1^0 = -\frac{1}{4}\sum_{k=0}^{3}\frac{1}{E_k} + \frac{W}{y_{0,+}}\frac{\vartheta_1'(\Delta_0-\Delta_\infty|\tau)}{\vartheta_1(\Delta_0-\Delta_\infty|\tau)} \tag{5.10}$$

$$= -\frac{W}{y_{0,+}}\left\{\frac{\vartheta_1'(\Delta_0+\Delta_\infty|\tau)}{\vartheta_1(\Delta_0+\Delta_\infty|\tau)} - \frac{\vartheta_1'(2\Delta_0|\tau)}{\vartheta_1(2\Delta_0|\tau)}\right\}, \tag{5.11}$$

$$\omega_1^\infty = \frac{1}{4}\sum_{k=0}^{3}E_k + W\frac{\vartheta_1'(\Delta_\infty-\Delta_0|\tau)}{\vartheta_1(\Delta_\infty-\Delta_0|\tau)} \tag{5.12}$$

$$= W\left\{\frac{\vartheta_1'(\Delta_\infty+\Delta_0|\tau)}{\vartheta_1(\Delta_\infty+\Delta_0|\tau)} - \frac{\vartheta_1'(2\Delta_\infty|\tau)}{\vartheta_1(2\Delta_\infty|\tau)}\right\}. \tag{5.13}$$

The constants Δ_0 and Δ_∞ are given by

$$\Delta_0 = \int_{(E_0,0)}^{P_{0,+}} \mathrm{d}v, \quad \Delta_\infty = \int_{(E_0,0)}^{P_{\infty,+}} \mathrm{d}v, \tag{5.14}$$

where $\mathrm{d}v$ is the normalized holomorphic differential $W\mathrm{d}z/y$ and W is the normalizing constant given by the equation

$$W\oint_{\mathfrak{a}} \frac{\mathrm{d}z}{y} = 1.$$

[3]We shall follow to the standard notation of the theory of elliptic functions fixed in [6].

Introduce the linear function of x, t

$$\mathfrak{L}(x,t) \equiv \int\limits_{(E_0,0)}^{\hat{\mu}(x,t)} \frac{\mathrm{d}z}{y} = 2i\,W\left(x + \frac{1}{y_{0,+}}t\right), \tag{5.15}$$

where $y_{0,+} = \sqrt{E_0E_1E_2E_3}$ and $\hat{\mu}(x,t) = (\mu(x,t), y(\mu(x,t)))$ with $y(\mu)^2 = R(\mu)$. This form of $\mathfrak{L}$ follows from the equations

$$\frac{\partial \mu}{\partial x} = 2iy(\mu), \quad \frac{\partial \mu}{\partial t} = -\frac{2i}{r^2}y(\mu). \tag{5.16}$$

It was already shown (see the end of Section 4) that the Thirring model represents a hidden form of the addition theorem for the Weierstrass ζ-function. In fact, the direct substitution of the solution into the first of the equations (1.1) leads to the equality

$$\begin{aligned} -\omega_1^\infty + W\frac{\vartheta_1'(\mathfrak{L}-\Delta_0)}{\vartheta_1(\mathfrak{L}-\Delta_0)} - W\frac{\vartheta_1'(\mathfrak{L}-\Delta_\infty)}{\vartheta_1(\mathfrak{L}-\Delta_\infty)} + \mathrm{e}^{\omega_0^{0,+}-\omega_0^{\infty,-}}\frac{\vartheta_1(\mathfrak{L}+\Delta_\infty)\vartheta_1(\mathfrak{L}-\Delta_\infty)}{\vartheta_1(\mathfrak{L}+\Delta_0)\vartheta_1(\mathfrak{L}-\Delta_0)} \\ + \mathrm{e}^{\omega_0^{\infty,+}-\omega_0^{\infty,-}}\frac{\vartheta_1(\mathfrak{L}+\Delta_\infty)\vartheta_1(\mathfrak{L}-2\Delta_\infty+\Delta_0)}{\vartheta_1(\mathfrak{L}+\Delta_0)\vartheta_1(\mathfrak{L}-\Delta_\infty)} = 0. \end{aligned} \tag{5.17}$$

The first three terms, after the substitution of the expression ω_1^∞ and using standard expressions for the Weierstrass ζ-functions, take the form

$$-\frac{1}{4}\sum_{k=0}^{3}E_k + 2W\omega\{\zeta(2\omega(\Delta_0-\Delta_\infty)) + \zeta(2\omega(\mathfrak{L}-\Delta_0)) - \zeta(2\omega(\mathfrak{L}-\Delta_\infty))\}.$$

The fourth term is an elliptic function and can be expressed in terms of the Weierstrass ζ-function as follows

$$\begin{aligned} &\mathrm{e}^{\omega_0^{0,+}-\omega_0^{\infty,-}}\frac{\vartheta_1(\mathfrak{L}+\Delta_\infty|\tau)\vartheta_1(\mathfrak{L}-\Delta_\infty|\tau)}{\vartheta_1(\mathfrak{L}+\Delta_0|\tau)\vartheta_1(\mathfrak{L}-\Delta_0)|\tau} \\ &= 2W\omega\{\zeta(2\omega(\mathfrak{L}+\Delta_0)) - \zeta(2\omega(\mathfrak{L}-\Delta_0)) - \zeta(2\omega(\Delta_\infty+\Delta_0)) + \zeta(2\omega(\Delta_\infty-\Delta_0))\}. \end{aligned}$$

Let us prove this equality. The left-hand side has the form

$$\frac{W\vartheta_1'(0|\tau)\vartheta_1(2\Delta_0|\tau)}{\vartheta_1(\Delta_0+\Delta_\infty|\tau)\vartheta_1(\Delta_0-\Delta_\infty|\tau)}\frac{\vartheta_1(\mathfrak{L}+\Delta_\infty|\tau)\vartheta_1(\mathfrak{L}-\Delta_\infty)|\tau}{\vartheta_1(\mathfrak{L}+\Delta_0|\tau)\vartheta_1(\mathfrak{L}-\Delta_0|\tau)}.$$

This function is doubly periodic with periods 1 and τ and has two first order poles at the points $\pm\Delta_0$ with residues $\pm W$, and first order zeros at the points $\pm\Delta_\infty$. Therefore it can be represented in the form

$$\zeta(2\omega(\mathfrak{L}+\Delta_0)) - \zeta(2\omega(\mathfrak{L}-\Delta_0)) + C,$$

where the constant C must be chosen to provide zeros at the points $\pm\Delta_\infty$. Because of the parity properties of the ζ-function, one value of the constant C serves for both points Δ_∞.

Analogously we can present the last term in the form

$$\begin{aligned} &\mathrm{e}^{\omega_0^{\infty,+}-\omega_0^{\infty,-}}\frac{\vartheta_1(\mathfrak{L}+\Delta_\infty|\tau)\vartheta_1(\mathfrak{L}-2\Delta_\infty+\Delta_0|\tau)}{\vartheta_1(\mathfrak{L}+\Delta_0|\tau)\vartheta_1(\mathfrak{L}-\Delta_\infty|\tau)} \\ &\quad = 2W\omega\{\zeta(2\omega(\mathfrak{L}-\Delta_\infty)) - \zeta(2\omega(\mathfrak{L}+\Delta_0)) + \zeta(4\omega\Delta_0) - \zeta(2\omega(\Delta_\infty-\Delta_0)\}. \end{aligned}$$

To prove this equality we shall write the left-hand side of the equality

$$\frac{W\vartheta_1'(0|\tau)\vartheta_1(\Delta_0+\Delta_\infty|\tau)}{\vartheta_1(2\Delta_\infty|\tau)\vartheta_1(\Delta_0-\Delta_\infty|\tau)}\frac{\vartheta_1(\mathfrak{L}-2\Delta_\infty+\Delta_0|\tau)\vartheta_1(\mathfrak{L}-\Delta_0|\tau)}{\vartheta_1(\mathfrak{L}+\Delta_0|\tau)\vartheta_1(\mathfrak{L}-\Delta_\infty|\tau)}.$$

This function has first order zeros at the points $-\Delta_\infty$, Δ_0 and first order poles at the points Δ_∞, $-\Delta_0$ with residues W and W correspondingly and can be therefore represented in the form

$$2W\omega\{-\zeta(2\omega(\mathfrak{L}-\Delta_\infty))+\zeta(2\omega(\mathfrak{L}+\Delta_0))+C\},$$

where the constant C is chosen to provide the vanishing properties of this function. The sum of all these expressions reduces to the equality (4.15), which was already proved.

5.2. **Further restriction of the elliptic curve.** Let us specify further the elliptic solution and give it in terms of standard elliptic functions. Suppose that the points $E_0,\ldots,E_3$ are placed symmetrically on the circle of the radius r, i.e.,

$$E_0=r\mathrm{e}^{i\theta},\quad E_1=r\mathrm{e}^{-i\theta},\quad E_2=r\mathrm{e}^{i\phi},\quad E_3=r\mathrm{e}^{-i\phi}, \tag{5.18}$$

where θ, ϕ is real and $0\le\theta<\phi\le\pi$.

The substitution

$$\xi=i\ \cot\frac{\phi}{2}\,\frac{r-z}{r+z} \tag{5.19}$$

transforms the curve (5.3) to the Legendre form

$$\widetilde{\mathcal{K}}_1=\{(\nu,\xi)\mid \nu^2=(1-\xi^2)(1-\tilde{k}^2\xi^2)\},\quad \tilde{k}=\tan\frac{\phi}{2}\cot\frac{\theta}{2}, \tag{5.20}$$

with the holomorphic differential

$$W\frac{\mathrm{d}z}{y}=-\frac{1}{4\widetilde{K}}\frac{\mathrm{d}\xi}{\nu},$$

where $\tilde{k}$ is the Jacobian module, and the normalizing constant is given by

$$W=-\frac{ir}{2\widetilde{K}}\cos\frac{\phi}{2}\sin\frac{\theta}{2}. \tag{5.21}$$

To find the substitution (5.19), following [49, §22.71–72], we write the holomorphic differential in the form

$$\frac{\mathrm{d}z}{y}=\frac{\mathrm{d}z}{\sqrt{[\cos^2\frac{\theta}{2}(z-r)^2+\sin^2\frac{\theta}{2}(z+r)^2][\cos^2\frac{\phi}{2}(z-r)^2+\sin^2\frac{\phi}{2}(z+r)^2]}}.$$

To make further computations more explicit we shall display the chosen dissection of the Riemann surface of the curve (5.3) in Fig. 2 and the map of the basis cycles under the map (5.19) in Fig. 3.

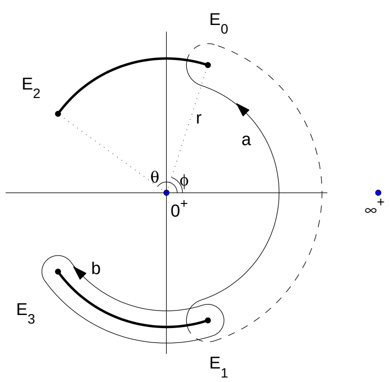

FIGURE 2. Basis of cycles of the elliptic complex curve with branching points (5.18).

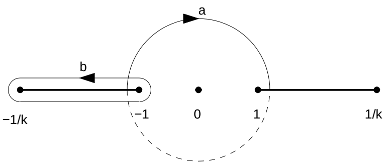

FIGURE 3. Basis of cycles after the transformation (5.19).

The Jacobian parameter $\tilde{\tau}$ is computed as

$$\begin{aligned}\tilde{\tau} &= \int_b \omega = -\frac{1}{2\widetilde{K}}\int_{-1}^{-\frac{1}{k}}\frac{\mathrm{d}\xi}{\nu}\\ &= -\frac{1}{2\widetilde{K}}\left\{\int_{-1}^{0}\frac{\mathrm{d}\xi}{\nu}+\int_{0}^{-\frac{1}{k}}\frac{\mathrm{d}\xi}{\nu}\right\} = -\frac{1}{2\widetilde{K}}\left\{\int_{0}^{1}\frac{\mathrm{d}\xi}{\nu}-\int_{0}^{\frac{1}{k}}\frac{\mathrm{d}\xi}{\nu}\right\}\\ &= -\frac{1}{2\widetilde{K}}\left\{\tilde{K}-\frac{1}{\tilde{k}}\tilde{K}\left(\frac{1}{\tilde{k}}\right)\right\} = \frac{i\widetilde{K}'}{2\widetilde{K}} = \frac{1}{2}\tau\end{aligned}$$

and therefore the transformation of the second order (the Gauss transformation, [6]) links the elliptic functions of the curves (5.3) and (5.20). The moduli are connected as follows

$$k = \frac{2\tilde{k}^{1/2}}{1+\tilde{k}}, \quad k' = \sqrt{\frac{1-\tilde{k}}{1+\tilde{k}}}, \quad K = (1+\tilde{k})\widetilde{K},\ K' = \frac{1}{2}(1+\tilde{k})\widetilde{K}'.$$

In what follows we shall use the standard formulas of the Gauss map.

One can compute

$$\int_{Q_0}^{P_{0\pm}} \omega = \pm\left(\frac{1}{4}-f\right), \quad \int_{Q_0}^{P_{\infty\pm}} \omega = \pm\left(\frac{1}{4}+f\right),$$

where $f = F(i\cot\frac{\phi}{2};\tilde{k})/4\widetilde{K}$, and $F(\,\cdot\,;\tilde{k})$ is the incomplete elliptic integral of the first kind. More precisely $\xi(0) = i\cot\frac{\phi}{2}$, while $\xi(\infty) = -i\cot\frac{\phi}{2}$. Then

$$\begin{aligned}\int_{Q_0}^{P_{0,+}} \omega &= -\frac{1}{4\widetilde{K}}\int_{1}^{i\cot\frac{\phi}{2}}\frac{\mathrm{d}\xi}{\nu}\\ &= -\frac{1}{4\widetilde{K}}\left(-K+\int_{0}^{i\cot\frac{\phi}{2}}\frac{\mathrm{d}\xi}{\nu}\right) = \frac{1}{4}-f.\end{aligned}$$

Analogously,

$$\begin{aligned}\int_{Q_0}^{P_{\infty,+}} \omega &= -\frac{1}{4\widetilde{K}}\int_{1}^{-i\cot\frac{\phi}{2}}\frac{\mathrm{d}\xi}{\nu} = -\frac{1}{4\widetilde{K}}\left(-K+\int_{0}^{-i\cot\frac{\phi}{2}}\frac{\mathrm{d}\xi}{\nu}\right)\\ &= \frac{1}{4}-\frac{1}{4\widetilde{K}}F(-i\cot\frac{\phi}{2};\tilde{k}) = \frac{1}{4}+f\end{aligned}$$

Let us show that (5.11), (5.13) are proved by the addition theorem for elliptic functions.

Proposition 5.1. *The equalities* (5.11), (5.13) *are proved by the addition theorem for elliptic functions.*

Proof. Consider (5.11) and write it in the form

$$2\omega W\{\zeta(4\omega\Delta_0)-\zeta(2\omega(\Delta_0+\Delta_\infty))-\zeta(2\omega(\Delta_0-\Delta_\infty))=\frac{r}{2}(\cos\theta+\cos\phi)\},$$

where ζ is the Weierstrass ζ-function with quasi-periods 2ω, $2\omega'$ [4] and parameters

$$\begin{aligned}
e_1&=-\frac{r^2}{6}-\frac{r^2}{6}\cos(\phi+\theta)+\frac{r^2}{3}\cos(\phi-\theta),\\
e_2&=\frac{r^2}{3}-\frac{r^2}{6}\cos(\phi+\theta)-\frac{r^2}{6}\cos(\phi-\theta),\\
e_3&=-\frac{r^2}{6}+\frac{r^2}{6}\cos(\phi+\theta)-\frac{r^2}{3}\cos(\phi-\theta).
\end{aligned}$$

The substitution of (5.21) and $w=K/\sqrt{e_1-e_3}$ and an application of the Weierstrass addition theorem for ζ-functions leads to the equality

$$\frac{i(1+\tilde{k})}{\sqrt{e_1-e_3}}\frac{\wp'(2\omega(\Delta_0+\Delta_\infty))-\wp'(2\omega(\Delta_0-\Delta_\infty))}{\wp(2\omega(\Delta_0+\Delta_\infty))-\wp(2\omega(\Delta_0-\Delta_\infty))}\frac{\cos(\theta)+\cos(\phi)}{\cos\frac{\phi}{2}\sin\frac{\theta}{2}}. \tag{5.22}$$

Because $\Delta_0+\Delta_\infty=\frac{1}{2}$, $\Delta_0-\Delta_\infty=-2f$ the left-hand side of the equality (5.22) is transformed as follows

$$\begin{aligned}
\mathrm{LHS}&=\frac{i(1+\tilde{k})}{\sqrt{e_1-e_3}}\frac{\wp'(4\omega f)}{e_1-\wp(4\omega f)}\\
&=2i(1+\tilde{k})\sqrt{\frac{(\wp(4\omega f)-e_2)\,(\wp(4\omega f)-e_3)}{(\wp(4\omega f)-e_1)\,(e_1-e_3)}}\\
&=2i(1+\tilde{k})\frac{\mathrm{dn}[\mathfrak{Z};k]}{\mathrm{cn}[\mathfrak{Z};k]\mathrm{sn}[\mathfrak{Z};k]},
\end{aligned}$$

where the argument and module of the Jacobian elliptic functions, $[\mathfrak{Z};k]$ read

$$[\mathfrak{Z};k]=\left[(1+\tilde{k})4\widetilde{K}f;\frac{2\tilde{k}^{\frac{1}{2}}}{1+\tilde{k}}\right].$$

The application of the Gauss transform reduces this expression to the form

$$2i\frac{1-\tilde{k}^2\mathrm{sn}^4[4\widetilde{K}f;\tilde{k}]}{\mathrm{sn}[4\widetilde{K}f;\tilde{k}]\mathrm{cn}[4\widetilde{K}f;\tilde{k}]\mathrm{dn}[4\widetilde{K}f;\tilde{k}]}.$$

By substituting

$$\mathrm{sn}[4\widetilde{K}f;\tilde{k}]=i\cot\frac{\phi}{2},\quad \mathrm{cn}[4\widetilde{K}f;\tilde{k}]=\frac{1}{\sin\frac{\phi}{2}},\quad \mathrm{dn}[4\widetilde{K}f;\tilde{k}]=\frac{1}{\sin\frac{\theta}{2}},$$

we obtain the right-hand side of the equality (5.22). □

Because of the formulas (5.4), (5.6), the following equality is valid

$$\frac{1}{r^2}\mu=\frac{u}{v}. \tag{5.23}$$

It is straightforward to show that

$$\mathfrak{L}(x,t)=\frac{1}{\widetilde{K}}(rx-\frac{1}{r}t-x_0)\cos\frac{\theta}{2}\sin\frac{\phi}{2}, \tag{5.24}$$

and

$$\mu=r\frac{i\cot\frac{\phi}{2}-\mathrm{sn}(4\widetilde{K}\mathfrak{L}(x,t);\tilde{k})}{i\cot\frac{\phi}{2}+\mathrm{sn}(4\widetilde{K}\mathfrak{L}(x,t);\tilde{k})}, \tag{5.25}$$

where (5.19) was used.

[4] In this context ω and ω' are the Weierstrass parameters.

Let us consider (5.23). First of all we want to prove the equality

$$e^{-\omega_0^{0,+}+\omega_0^{\infty,-}} = \frac{1}{r}. \tag{5.26}$$

We have:

$$\begin{aligned}
\frac{\vartheta_1(2f|\tau)}{\vartheta_2(2f|\tau)} &= k' \frac{\mathrm{sn}\left[\frac{K}{\widetilde{K}}F(i\cot\frac{\phi}{2};\tilde{k});k\right]}{\mathrm{cn}\left[\frac{K}{\widetilde{K}}F(i\cot\frac{\phi}{2};\tilde{k});k\right]}\\
&= \sqrt{\frac{1-\tilde{k}}{1+\tilde{k}}}\frac{\mathrm{sn}\left[(1+\tilde{k})F(i\cot\frac{\phi}{2};\tilde{k});\frac{2\tilde{k}^{1/2}}{1+\tilde{k}}\right]}{\mathrm{cn}\left[(1+\tilde{k})F(i\cot\frac{\phi}{2};\tilde{k});\frac{2\tilde{k}^{1/2}}{1+\tilde{k}}\right]}\\
&= \tilde{k}'\frac{\mathrm{sn}\left[F(i\cot\frac{\phi}{2};\tilde{k});\tilde{k}\right]}{\mathrm{cn}\left[F(i\cot\frac{\phi}{2};\tilde{k});\tilde{k}\right]\mathrm{dn}\left[F(i\cot\frac{\phi}{2};\tilde{k});\tilde{k}\right]} = i\tilde{k}'\sin\frac{\theta}{2}\cos\frac{\phi}{2},\\
\frac{\vartheta_1'(0|\tau)W}{\vartheta_2(0|\tau)} &= \frac{\pi r\vartheta_3(0|\tau)\vartheta_4(0|\tau)\sin\frac{\theta}{2}\cos\frac{\phi}{2}}{i2\widetilde{K}} = ir\tilde{k}'\sin\frac{\theta}{2}\cos\frac{\phi}{2}.
\end{aligned}$$

The expressions (5.8), (5.9), in combination with the above formulas, prove (5.26).

Because of the equality

$$\vartheta_1(y+z|\tau)\vartheta_1(y-z|\tau) = \vartheta_3(2y|2\tau)\vartheta_2(2z|2\tau) - \vartheta_2(2y|2\tau)\vartheta_3(2z|2\tau)$$

the ratio ψ_1/ψ_2 is equal to

$$\frac{1}{r}\frac{\vartheta_3(2\mathfrak{L}(x,t)|2\tau)\vartheta_2(2f-\frac{1}{2}|2\tau) - \vartheta_2(2\mathfrak{L}(x,t)|2\tau)\vartheta_3(2f-\frac{1}{2}|2\tau)}{\vartheta_3(2\mathfrak{L}(x,t)|2\tau)\vartheta_2(2f+\frac{1}{2}|2\tau) - \vartheta_2(2\mathfrak{L}(x,t)|2\tau)\vartheta_3(2f+\frac{1}{2}|2\tau)}.$$

Let $\mathfrak{L}(x,t) = \widetilde{\mathfrak{L}}(x,t) + \frac{1}{4}$. Then the above expression can be transformed to the form

$$\begin{aligned}
\frac{\psi_1}{\psi_2} &= -\frac{1}{r}\frac{\vartheta_4(2\widetilde{\mathfrak{L}}(x,t)|2\tau)\vartheta_1(2f|2\tau) + \vartheta_1(2\widetilde{\mathfrak{L}}(x,t)|2\tau)\vartheta_4(2f|2\tau)}{\vartheta_4(2\widetilde{\mathfrak{L}}(x,t)|2\tau)\vartheta_1(2f|2\tau) - \vartheta_1(2\widetilde{\mathfrak{L}}(x,t)|2\tau)\vartheta_4(2f|2\tau)} &(5.27)\\
&= \frac{1}{r}\frac{\mathrm{sn}(4\widetilde{K}f;\tilde{k}) + \mathrm{sn}(4\widetilde{K}\widetilde{\mathfrak{L}}(x,t);\tilde{k})}{\mathrm{sn}(4\widetilde{K}f;\tilde{k}) - \mathrm{sn}(4\widetilde{K}\widetilde{\mathfrak{L}}(x,t);\tilde{k})}, &(5.28)
\end{aligned}$$

which coincides with (5.25).

Taking into the account the expressions for the above constants, we write finally the elliptic solution of the Thirring model associated with the branch points (5.18) of the curve in the following form

$$u(x,t) = \frac{2}{\sqrt{r}}\sin\frac{\theta}{2}\cos\frac{\phi}{2}\exp(-2i(\omega_1^\infty x - \omega_1^0 t))\frac{\vartheta_1\left(\mathfrak{L}(x,t)-\frac{1}{4}+f|\tau\right)}{\vartheta_1\left(\mathfrak{L}(x,t)-\frac{1}{4}-f|\tau\right)}, \tag{5.29}$$

$$v(x,t) = 2\sqrt{r}\sin\frac{\theta}{2}\cos\frac{\phi}{2}\exp(-2i(\omega_1^\infty x - \omega_1^0 t))\frac{\vartheta_1\left(\mathfrak{L}(x,t)+\frac{1}{4}+f|\tau\right)}{\vartheta_1\left(\mathfrak{L}(x,t)+\frac{1}{4}-f|\tau\right)}, \tag{5.30}$$

where ω_1^∞, ω_1^0 are given in (5.13), (5.11).

5.3. **The soliton limit.** Now we are in the position to discuss the soliton limit,

$$\theta \to \phi, \quad k \to 1, \quad K \to \infty, \quad K' = \frac{1}{2}i\pi. \tag{5.31}$$

In this limit the expression (5.28) is reduced to the form

$$\frac{u}{v} = \frac{1}{r}\frac{\cosh\left[2\sin\phi\,(rx-\frac{1}{r}t-x_0)-\frac{i}{2}\phi\right]}{\cosh\left[\sin\phi\,(rx-\frac{1}{r}t-x_0)+\frac{i}{2}\phi\right]}, \tag{5.32}$$

which is also in accordance with the Kuznetsov–Mikhailov formula (5.2).

Furthermore, to compute the exponentials we find

$$\omega_1^\infty \longrightarrow 2r\cos\phi, \quad \omega_1^0 \longrightarrow -\frac{2}{r}\cos\phi. \tag{5.33}$$

Let us derive the first of these formulas. To do that we must compute the asymptotic of the θ-function part of the formula (5.13). We have

$$\begin{aligned} W\frac{\vartheta_1'(\Delta_\infty-\Delta_0)}{\vartheta_1(\Delta_\infty-\Delta_0)} &= 2\omega W\left\{\zeta(2\omega(\Delta_\infty-\Delta_0)) - \frac{\eta}{\omega}2\omega(\Delta_\infty-\Delta_0)\right\} \\ &= 2\omega W\left\{\zeta(u) - \frac{\eta}{\omega}u\right\}, \end{aligned}$$

where

$$u = 4f\omega = \frac{1+\tilde{k}}{\sqrt{e_1-e_3}}F(i\cot\frac{\phi}{2}).$$

In the soliton limit

$$\begin{aligned} u &\longrightarrow i\frac{\pi-\phi}{\sqrt{e_1-e_3}} \\ \zeta(u) - \frac{\eta}{\omega}u &\longrightarrow \sqrt{e_1-e_3}\tanh u, \end{aligned}$$

where the formula $E = (e_1\omega+\eta)/\sqrt{e_1-e_3}$ was used to compute the second asymptotic. Taking into account that

$$2\omega W = \frac{ir(1+\tilde{k})\cos\frac{\phi}{2}\sin\frac{\theta}{2}}{\sqrt{e_1-e_3}},$$

we obtain the required result.

Direct computation of this limit of the θ-ratios on the basis of known asymptotic θ-function formulas is difficult because in this limit the θ-function diverges ($\tau\to 0$) while the arguments of both θ-functions tend to $\pm\frac{1}{4}$. To overcome this problem we shall use the explicit expression for the differentials of the third kind

$$\mathrm{d}\omega_{P_{0_\mp},P_{\infty_\mp}} = \mathrm{d}\tilde{\omega}_{P_{0_\mp},P_{\infty_\mp}} \mp \frac{\lambda_1\mathrm{d}z}{2y}.$$

First we remark that comparing (4.8), (4.9) and (5.11), (5.13), one is led to the conclusion that

$$\Lambda_1 = W\frac{\vartheta_1'(\Delta_0-\Delta_\infty|\tau)}{\vartheta_1(\Delta_\infty-\Delta_0|\tau)}.$$

In the soliton limit we have

$$\Lambda_1 \longrightarrow 2r\cos(\phi).$$

The non-normalized part of the differential of the third kind can be written as follows

$$\mathrm{d}\tilde{\omega}_{P_{0_\mp},P_{\infty_\mp}} = \frac{y+y_{0,\mp}}{2z}\frac{\mathrm{d}z}{y} \pm \frac{z\mathrm{d}z}{2y} \tag{5.34}$$

$$= \frac{i\cot(\frac{\phi}{2})}{\xi^2+\cot^2(\frac{\phi}{2})}\left\{1 \pm \frac{1}{i\cos\frac{\phi}{2}\sin\frac{\theta}{2}}\frac{\xi}{\sqrt{(1-\xi^2)(1-\tilde{k}^2\xi)}}\right\}\mathrm{d}\xi. \tag{5.35}$$

The second line in this formula results from the substitution of (5.19). The differentials are normalized which follows from the comparison of (4.8), (4.9) and (5.11), (5.13).

It is straightforward to show by substituting $\xi = \tanh(\mathcal{M})$, $\tilde{k} = 1$, that in the soliton limit

$$\mathrm{d}\omega_{P_{0_\mp},P_{\infty_\mp}} \longrightarrow \cosh^2(\mathcal{M})\tanh(\mathcal{M}\pm i\frac{\phi}{2})\mathrm{d}\xi = \tanh(\mathcal{M}\pm i\frac{\phi}{2})\mathrm{d}\mathcal{M}.$$

Evidently

$$\mathrm{d}_\mu \ln \frac{\vartheta_1\left(\mathcal{L}(x,t) \mp \frac{1}{4} + f|\tau\right)}{\vartheta_1\left(\mathcal{L}(x,t) \mp \frac{1}{4} - f|\tau\right)} = \pm \mathrm{d}_\mu \ln \frac{\vartheta_1\left(\int_{E_0}^{\mu} \omega \pm \Delta_0|\tau\right)}{\vartheta_1\left(\int_{E_0}^{\mu} \omega \pm \Delta_\infty|\tau\right)} = \mathrm{d}\omega_{P_{0_\mp}, P_{\infty_\mp}}.$$

But the logarithmic derivative of the hyperbolic function in the Kuznetsov–Mikhailov formula reads

$$\mathrm{d}_{\mathcal{M}} \ \ln \frac{1}{\cosh(\mathcal{M} \pm \frac{i\phi}{2})} = \tanh(\mathcal{M} \pm i\frac{\phi}{2})\mathrm{d}\mathcal{M}.$$

This last equality completes the proof of (5.1) and (5.2) from (5.29) and (5.30).

Acknowledgments. We are indepted to Fritz Gesztesy for helpful comments. The authors are grateful to Dmitry Leykin for careful discussion of the Baker ζ-formula (3.20) and information on the general expression in matrix form for arbitrary genus for the set of formulas (3.21).

References

[1] S Abenda and Yu Fedorov. On the weak Kowalevski-Painlevé property for hyperelliptic separable systems. *Acta Appl. Math.*, 60:137–178, 2000.

[2] H F Baker. *Abelian Functions: Abel's Theorem and the Allied Theory Including the Theory of Theta Functions.* Cambridge Univ. Press, Cambridge, 1897, reprinted 1995.

[3] H F Baker. On the hyperelliptic sigma functions. *Amer. Journ. Math.*, 20:301–384, 1898.

[4] H F Baker. On a system of differential equations leading to periodic functions. *Acta Math.*, 27:135–156, 1903.

[5] H F Baker. *Multiply Periodic Functions.* Cambridge Univ. Press, Cambridge, 1907.

[6] H Bateman and A Erdelyi. *Higher Transcendental Functions*, volume 2. McGraw-Hill, New York, 1955.

[7] E D Belokolos and V Z Enolskii. Generalized Lamb ansatz. *Theoret. and Math. Phys.*, 53:1120–1127, 1982.

[8] R F Bikbaev. Finite-gap solutions of the massive Thirring model. *Theoret. and Math. Phys.*, 63(3): 577–584, 1985.

[9] O Bolza. On the first and second logarithmic derivatives of hyperelliptic σ–functions. *Amer. Journ. Math.*, 17:11–36, 1895.

[10] O Bolza. The partial differential equations for the hyperelliptic θ and σ-functions. *Amer. Journ. Math.*, 21:107–125, 1899.

[11] O Bolza. Proof of Brioschi's recursion formula for the expansions of the even σ-functions of two variables. *Amer. Journ. Math.*, 21:175–190, 1899.

[12] O Bolza. Remark concerning expansions of the hyperelliptic σ-functions. *Amer. Journ. Math.*, 22:101–112, 1900.

[13] V M Buchstaber, J C Eilbeck, V Z Enolskii, D V Leykin, and M Salerno. Multidimensional Schrödinger equation with Abelian potential. *J. Math. Phys.*, 43:2858–2881, 2002.

[14] V M Buchstaber, V Z Enolskii, and D V Leykin. *Hyperelliptic Kleinian functions and applications*, volume 179, pages 1–34. Advances in Math. Sciences, AMS Translations, series – 2, Moscow State University and University of Maryland, College Park, 1997.

[15] V M Buchstaber, V Z Enolskii, and D V Leykin. Kleinian functions, hyperelliptic Jacobians and applications. In S P Novikov and I M Krichever, editors, *Reviews in Mathematics and Mathematical Physics*, volume 10:2, pages 1–125, London, 1997. Gordon and Breach.

[16] V M Buchstaber, V Z Enolskii, and D V Leykin. Recursive family of polynomials generated by Sylvester's identity and addition theorem for hyperelliptic Kleinian functions. *Func. Anal. Appl.*, 31(4):19–32, 1997.

[17] V M Buchstaber, V Z Enolskii, and D V Leykin. Rational analogues of the abelian functions. *Func. Anal. Appl.*, 33(2):1–15, 1999.

[18] V M Buchstaber, V Z Enolskii, and D V Leykin. Uniformisation of Jacobi varieties of trigonal curves and nonlinear differential equations. *Func. Anal. Appl.*, 34(3):1–15, 2000.

[19] V M Buchstaber and D V Leykin. Lie algebras associated with sigma functions and versal deformations. *Uspekhi Math. Nauk*, 57(3):145–146, 2002. To appear in Russ. Math. Surv.

[20] V M Buchstaber and D V Leykin. Graded Lie algebras that define hyperelliptic sigma functions. *Dokl. Akad. Nauk*, 385(5), 2002. English translation in Dokl. Math. Sci 66(4/2), 2002.

[21] H Burkhardt. Beiträge zur Theorie der hyperelliptische Sigmafunktionen. *Math. Ann.*, 32:381–442, 1888.

[22] P L Christiansen, J C Eilbeck, V Z Enolskii, and N A Kostov. Quasi periodic solutions of Manakov type coupled nonlinear Schrödinger equations. *Proc. R. Soc. Lond. A*, 456:2263–2281, 2000.

[23] E Date. On quasi-periodic solutions of the field equation of the classical massive Thirring model. *Progr. Theor. Phys.*, 59:265–273, 1978.

[24] J D Edelstein, M Gómez-Reino, and M Mariño. Blowup formulas in Donaldson-Witten theory and integrable hierarchies. *Adv. Theor. Math. Phys.* 4:503–543, 2000.

[25] J C Eilbeck, V Z Enolskii, and N A Kostov. Quasi periodic solutions for vector nonlinear Schrödinger equations. *J. Math. Phys.* 41:8236–8248, 2000.

[26] J C Eilbeck, V Z Enolskii, and D V Leykin. On the Kleinian construction of Abelian functions of canonical algebraic curve. In *Proceedings of the Conference SIDE III: Symmetries of Integrable Differences Equations, Saubadia, May 1998, CRM Proceedings and Lecture Notes*, pages 121–138, 2000.

[27] V Z Enolskii, F Gesztesy, and H Holden. The classical massive Thirring system revisited. In F Gesztesy, H Holden, J Jost, S Paycha, M Röckner, and S Scarlatti, editors, *Stochastic Processes, Physics and Geometry: New Interplays. I. A Volume in Honor of Sergio Albeverio, Canadian Mathematical Society Conference Proceeding Series*, Providence, RI, USA, pages 163–200, 2000.

[28] H M Farkas and I Kra. *Riemann Surfaces.* Springer, New York, 1980.

[29] J D Fay. *Theta Functions on Riemann Surfaces. Lectures Notes in Mathematics*, volume 352, Berlin, 1973. Springer.

[30] F Gesztesy and H Holden. *Soliton Equations and Their Algebro-Geometric Solutions. Vol. I: $(1+1)$-Dimensional Continuous Models.* Cambridge University Press, Cambridge, 2003.

[31] G Göpel. Theoriae transcendentium Abelianarum primi ordinis adumbratio levis. *Journ. reine angew. Math.*, 35(4):277-312, 1847.

[32] D Grant. A generalization of a formula of Eisenstein. *Proc. London Math. Soc.*, 62:121–132, 1991.

[33] P I Holod and A K Prikarpatsky. Classical solutions of two-dimensional Thirring model with periodic initial conditions. Preprint of the Institute of Theoretical Physics, 1978.

[34] A M Kamchatnov, H Steudel, and A A Zabolotskii. The Thirring model as an approximation to the theory of two-photon propagation. *J. Phys. A: Math. Gen*, 30:7485–7499, 1997.

[35] F Klein. Über hyperelliptische Sigmafunctionen. *Math. Ann.*, 27:431–464, 1886.

[36] F Klein. Über hyperelliptische Sigmafunctionen. *Math. Ann.*, 32:351–380, 1888.

[37] E A Kuznetsov and A V Mikhailov. On the complete integrability of the two-dimensional classical Thirring model. *Theor. and Math. Phys.*, 30:193–200, 1977.

[38] S Matsutani. Hyperelliptic solutions of KdV and KP equations: Reevaluation of Baker's study on hyperelliptic sigma functions. *J. Phys. A: Math. Gen.*, 34:4721–4732, 2001.

[39] S Matsutani and Y Ônishi. On the moduli of a quantized elastica in $\mathbb{P}$ and KdV flows: Study of hyperelliptic curves as an extention of Euler's perspective of elastica I. Preprint, 2000.

[40] A B Mikhailov. Integrability of the two-dimensional Thirring model. *JETP Lett.*, 23:320–323, 1976.

[41] Y Ônishi. Complex multiplication formulas for hyperelliptic curve of genus three. *Tokyo J. Math.*, 21(2):381–431, 1998.

[42] Y Ônishi. Determinant expressions for some Abelian functions in genus two. *Glasgow Math. J.*, 43: 2002 (To appear.).

[43] Y Ônishi. Determinant Expressions for Hyperelliptic Abelian Functions (with an Appendix by Shigeki Matsutani). *Preprint NT/0105189*, 2002.

[44] S M Roman. *The Umbral Calculus.* Academic Press, New York, 1984.

[45] G Rosenhain. Abhandlung uber die Funktionen zweier Variabler mit vier Perioden. *Mem. pres. l'Acad de Sci. de France des savants*, IX:361–455, 1851.

[46] W E Thirring. A solvable relativistic field theory. *Ann. of Phys.*, 3:91–112, 1958.

[47] K Weierstrass. Beitrag zur Theorie der Abel'schen Integrale. *Jahreber. Königl. Katolischen Gymnasium zu Braunsberg in dem Schuljahre 1848/49*, pages 3–23, 1849.

[48] K Weierstrass. Zur Theorie der Abelschen Functionen. *Journ. reine angew. Math.*, 47:289–306, 1854.

[49] E T Whittaker and G N Watson. *A Course of Modern Analysis.* Cambridge Univ. Press, Cambridge, 1973.

[50] E Wiltheiss. Ueber die Potenzreihen der hyperelliptischen Thetafunktionen. *Math. Ann.*, 31:410–423, 1888.

[51] M A Wisse. Darboux coordinates and isospectral Hamiltonian flows. *Lett. Math. Phys.*, 28:287–294, 1993.

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