IDENTITIES FOR THE CLASSICAL GENUS TWO $\wp$ FUNCTION.

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Abstract. We present a simple method that allows one to generate and classify identities for genus two $\wp$ functions for generic algebraic curves of type (2,6). We discuss the relation of these identities to the Boussinesq equation for shallow water waves and show, in particular, that these $\wp$ functions give rise to a family of solutions to Boussinesq.

1. Introduction

This paper is an introduction to the role of representation theory in the classical theory of the genus two $\wp$-function, the parametrizing function for the Jacobi variety associated with the algebraic curve

$$\mathcal{V}: \quad y^2 = g_6x^6 + 6g_5x^5 + 15g_4x^4 + 20g_3x^3 + 15g_2x^2 + 6g_1x + g_0.$$

Such a curve transforms under the map

$$\begin{aligned} x &\mapsto \frac{\alpha x+\beta}{\gamma x+\delta}, \\ y &\mapsto \frac{y}{(\gamma x+\delta)^3}, \end{aligned} \tag{1.1}$$

into a curve of the same kind but with different coefficients. In the classical treatment [2] such a transformation is chosen to make g_6 vanish and to normalise $6g_5$ to the value 4. The resulting canonical form,

$$\pi(\mathcal{V}): \quad Y^2 = 4X^5 + 15G_4X^4 + 20G_3X^3 + 15G_2X^2 + 6G_1X + G_0,$$

has a branch point at $X = \infty$. Note that this canonical form is not unique. There is still at least freedom under transformations (1.1) which would allow, say, G_4 to be set to the value zero. Note too that this canonical form does not cover *all* curves: for example, $y^2 = x^6$ does not have such a canonical form.

Holomorphic differentials of the first kind on the Jacobian variety $Symm(\pi(\mathcal{V})\otimes\pi(\mathcal{V}))$ are

$$\begin{aligned} dU_1 &= \frac{dX_1}{Y_1} + \frac{dX_2}{Y_2}, \\ dU_2 &= \frac{X_1dX_1}{Y_1} + \frac{X_2dX_2}{Y_2}, \end{aligned}$$

where (X_1, Y_1) and (X_2, Y_2) are analytic points on $\pi(\mathcal{V})$.

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Three (Kleinian) doubly indexed objects are defined:

$$
\begin{aligned}
P^K_{22} &= X_1 + X_2,\\
P^K_{12} &= -X_1X_2,\\
P^K_{11} &= \frac{F(X_1,X_2) - 2Y_1Y_2}{4(X_1-X_2)^2},
\end{aligned}
\tag{1.2}
$$

where $F(X_1, X_2)$ is the polar form of the quintic

$$
\begin{aligned}
F(X_1,X_2) = 4(X_1+X_2)X_1^2X_2^2 + 30G_4X_1^2X_2^2 + 20G_3(X_1+X_2)X_1X_2 +\\
30G_2X_1X_2 + 6G_1(X_1+X_2) + 2G_0.
\end{aligned}
$$

The notation $\wp$ is usual for these objects but we wish to reserve this symbol for covariant objects. In fact the classical treatments like [2] are slightly confusing in that $\wp$ is used for both classes of object except where the distinction is paramount. We will be using the notation of Art.13 of [2]. The superscript K is not conventional either but serves to distinguish these Kleinian objects from slightly different (Baker) P symbols to be introduced shortly.

It is then shown that $\partial_{U_1}P^K_{12} = \partial_{U_2}P^K_{11}$ and $\partial_{U_1}P^K_{22} = \partial_{U_2}P^K_{12}$, so that there exists a potential function P^K such that $P^K_{ij} = \partial_{U_i}\partial_{U_j}P^K$. This P^K function can be shown to satisfy numerous differential identities. In particular we have

$$
\begin{aligned}
P^K_{2222} - 6{P^K}^2_{22} &= 10G_3 + 15G_4P^K_{22} + 4P^K_{12},\\
P^K_{1222} - 6P^K_{22}P^K_{12} &= 15G_4P^K_{12} - 2P^K_{11},\\
P^K_{1122} - 2P^K_{22}P^K_{11} - 4{P^K}^2_{12} &= 10G_3P^K_{12},\\
P^K_{1112} - 6P^K_{12}P^K_{11} &= -G_0 - 3G_1P^K_{22} + 15G_2P^K_{12},\\
P^K_{1111} - 6{P^K}^2_{11} &= -\frac{15}{2}G_0G_4 + 15G_1G_3 - 3G_0P^K_{22},\\
&\quad +6G_1P^K_{12} + 15G_2P^K_{11},
\end{aligned}
\tag{1.3}
$$

where all subscripts are now interpreted as derivatives with respect to U_1 and U_2.

The two index objects also satisfy the important quartic relation

$$
\begin{vmatrix}
G_0 & 3G_1 & -2P^K_{11} & -2P^K_{12}\\
3G_1 & 4P^K_{11}+15G_2 & 2P^K_{12}+10G_3 & -2P^K_{22}\\
-2P^K_{11} & 2P^K_{12}+10G_3 & 4P^K_{22}+15G_4 & 2\\
-2P^K_{12} & -2P^K_{22} & 2 & 0
\end{vmatrix} = 0.
\tag{1.4}
$$

This last relation shows that the P^K_{ij} parametrise the Kummer variety and it is the starting point of the theory in Baker's treatment.

In treating the *generic* curve ($g_6 \neq 0, g_5 \neq \frac{2}{3}$) the definitions (1.2), with x and y replacing X and Y and with the polar form for the generic sextic, are *no longer* adequate because they do not give the correct transformation properties for the P^K_{ij} under the transformations (1.1) of x_1 and x_2. There are two ways round this problem.

The classical solution is to *force* the correct transformation properties by defining covariant $\wp$-functions, $\wp_{ij}$ in terms of the P^K_{ij} and the coefficients of the transformation (1.1) which takes the specific curve, $\mathcal{V}$, to its canonical form, $\pi(\mathcal{V})$. The current paper is devoted to the representation theory implicit in this.

A second solution to the problem is to define the $\wp$ functions in a different, covariant fashion right from the start. This approach is pursued in a separate publication [1].

For some recent applications of $\wp$ functions for hyperelliptic curves of general genus in the tradition of Baker's work, see [4, 6, 9, 10, 11].

2. The SL_2 action

In this section we consider the infinitesimal action on the space of curves associated with (1.1). In the next we construct the induced action on the space of canonical forms.

The curve $\mathcal{V}:\ y^2 = g(x)$ is to be thought of as a hypersurface in the nine dimensional complex space of variables and parameters $x, y, g_6, g_5, g_4, g_3, g_2, g_1, g_0$. The family of such hypersurfaces is permuted under the transformations (1.1) but the *covariance* of their form is expressed by the three conditions

$$\begin{aligned} e(y^2 - g(x)) &= 0, \\ h(y^2 - g(x)) &= -6(y^2 - g(x)), \\ f(y^2 - g(x)) &= 6x(y^2 - g(x)), \end{aligned}$$

where e, h and f are the vector fields

$$e = -\frac{\partial}{\partial x} + \sum_{p=0}^{6}(6-p)g_{p+1}\frac{\partial}{\partial g_p}, \tag{2.1}$$

$$h = -2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} + \sum_{p=0}^{6}(2p-6)g_p\frac{\partial}{\partial g_p}, \tag{2.2}$$

$$f = x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y} + \sum_{p=0}^{6} p g_{p-1}\frac{\partial}{\partial g_p}, \tag{2.3}$$

which form a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{2.4}$$

Holomorphic differentials for the generic curve are

$$\begin{aligned} du_1 &= \frac{dx_1}{y_1} + \frac{dx_2}{y_2}, \\ du_2 &= \frac{x_1 dx_1}{y_1} + \frac{x_2 dx_2}{y_2}, \end{aligned} \tag{2.5}$$

where the (x_i, y_i) are analytic points on the curve, and it is easy to see that these differentials transform simply under (1.1):

$$du_1 \mapsto \delta du_1 + \gamma du_2, \tag{2.6}$$

$$du_2 \mapsto \beta du_1 + \alpha du_2. \tag{2.7}$$

Consequently the derivatives, $\partial_i = \frac{\partial}{\partial u_i}$ for $i = 1, 2$ transform as

$$\partial_1 \mapsto \alpha\partial_1 - \beta\partial_2, \tag{2.8}$$

$$\partial_2 \mapsto -\gamma\partial_1 + \delta\partial_2, \tag{2.9}$$

and the action of e, f and h extend to first order derivatives thus

$$e(\partial_1) = \partial_2 \qquad e(\partial_2) = 0 \tag{2.10}$$

$$f(\partial_1) = 0 \qquad f(\partial_2) = \partial_1 \tag{2.11}$$

$$h(\partial_1) = -\partial_1 \qquad h(\partial_2) = \partial_2 \tag{2.12}$$

and to higher order derivatives via the Leibnitz rule, e.g.

$$e(\partial_1^3\partial_2^2) = 3\partial_1^2\partial_2^3.$$

3. The induced action on canonical forms and the actions on $\wp$ functions

Suppose now that the curve $\mathcal{V}$ is mapped to $\tilde{\mathcal{V}}$ under a map (1.1). These curves project down to canonical forms $\pi(\mathcal{V})$ and $\pi(\tilde{\mathcal{V}})$ so that there is an induced action of (1.1) on the canonical forms. The corresponding infinitesimal actions on the canonical forms will be denoted e^*, f^* and h^*.

The transformation π can be taken to be [2]

$$x = \frac{\mu X}{\frac{\mu}{\theta}X + \frac{1}{\mu}} \tag{3.1}$$

$$y = \frac{Y}{(\frac{\mu}{\theta}X + \frac{1}{\mu})^3} \tag{3.2}$$

where

$$g_6\theta^6 + 6g_5\theta^5 + 15g_4\theta^4 + 20g_3\theta^3 + 15g_2\theta^2 + 6g_1\theta + g_0 = 0$$

and

$$\frac{2}{3\mu^4} = g_5 + 5g_4/\theta + 10g_3/\theta^2 + 10g_2/\theta^3 + 5g_1/\theta^4 + g_0/\theta^5.$$

The parameters θ and μ must of course vary with the particular curve under consideration and therefore are themselves subject to the $\mathfrak{sl}_2(\mathbb{C})$ action of e, f and h. Application of these operators to the defining relations for θ and μ yields

$$e(\theta) = -1, \quad f(\theta) = \theta^2, \quad h(\theta) = -2\theta, \tag{3.3}$$

$$e(\mu) = -\frac{\mu}{\theta}, \quad f(\mu) = 0, \quad h(\mu) = -2\mu. \tag{3.4}$$

The e action is the infinitesimal form of the one parameter (t) subgroup of transformations (1.1)

$$x \mapsto x - t$$

and the induced action on the canonical variable X is

$$X \mapsto \frac{\mu^2\tilde{\theta}(\theta - t)X - \theta\tilde{\theta}t}{\mu^2\tilde{\mu}^2(t + \tilde{\theta} - \theta)X + \theta\tilde{\mu}^2(t + \tilde{\theta})}$$

where the tilded quantities appertain to the transformed curve and are therefore functions of t. In fact, for small t, $\tilde{\theta} = \theta - t + O(t^2)$, $\tilde{\mu} = \mu - \frac{\mu}{\theta}t + O(t^2)$, and so

$$X \mapsto X - \frac{1}{\mu^2}t + O(t^2).$$

The e action on Y follows by a similar argument and on the G_p by expressing them in terms of the g_p, θ and μ. We obtain

$$e^* = \frac{1}{\mu^2}E \tag{3.5}$$

where

$$E = -\frac{\partial}{\partial X} + \sum_{p=0}^{3}(6-p)G_{p+1}\frac{\partial}{\partial G_p} + \frac{4}{3}\frac{\partial}{\partial G_4}.$$

By precisely similar arguments (or by noting that π effectively factors out the one parameter subgroups generated by f and h) we find that

$$f^* = 0 \tag{3.6}$$

$$h^* = 0. \tag{3.7}$$

If we return now to the definitions of the two index objects, equations (1.2), we *might* expect that they should behave according to the rules for second order derivatives under the e^* action $e^*(P^K_{22}) = 0$, $e^*(P^K_{12}) = P^K_{22}$ and $e^*(P^K_{11}) = 2P^K_{12}$. But instead we find (directly from their definitions as functions of the X_i) that under the E action,

$$E(P^K_{22}) = -2, \quad E(P^K_{12}) = 2P^K_{22}, \quad E(P^K_{11}) = P^K_{12}, \tag{3.8}$$

which isn't quite correct. The situation is mollified by adding constants to the P^K_{ij}, to define new P functions [2], namely:

$$\begin{aligned} P_{22} &= P^K_{22} + \frac{3}{2}G_4 \\ P_{12} &= P^K_{12} + \frac{1}{2}G_3 \\ P_{11} &= P^K_{11} + \frac{3}{2}G_2. \end{aligned} \tag{3.9}$$

These functions satisfy the correct relations with respect to the operator E (but not e^*). Of course, there are no operators F and H.

Baker [2] defines the covariant ℘ functions by insisting that they transform from the P_{ij} as second derivatives. That is, he uses the maps

$$\begin{aligned} \partial_1^2 &\mapsto \alpha^2\partial_1^2 - 2\alpha\beta\partial_1\partial_2 + \beta^2\partial_2^2 \\ \partial_1\partial_2 &\mapsto -\alpha\gamma\partial_1^2 + 2(\alpha\delta+\beta\gamma)\partial_1\partial_2 - \beta\delta\partial_2^2 \\ \partial_2^2 &\mapsto \gamma^2\partial_1^2 - 2\gamma\delta\partial_1\partial_2 + \delta_2\partial_2^2 \end{aligned} \tag{3.10}$$

with the values $\alpha = \mu$, $\beta = 0$, $\gamma = \frac{\mu}{\theta}$ and $\delta = \frac{1}{\mu}$ borrowed from π, to define

$$\wp_{11} = \mu^2 P_{11} \tag{3.11}$$

$$\wp_{12} = -\frac{\mu^2}{\theta}P_{11} + P_{12} \tag{3.12}$$

$$\wp_{22} = \frac{\mu^2}{\theta^2}P_{11} - \frac{2}{\theta}P_{12} + \frac{1}{\mu^2}P_{22}. \tag{3.13}$$

These $\wp$ functions are now genuinely covariant as is easily checked by application of e and f. For example

$$\begin{aligned}
e(\wp_{12}) &= -e\left(\frac{\mu^2}{\theta}\right)P_{11} - \frac{\mu^2}{\theta}e^*(P_{11}) + e^*(P_{12}) \\
&= \frac{\mu^2}{\theta^2}P_{11} - \frac{1}{\theta}E(P_{11}) + \frac{1}{\mu^2}E(P_{12}) \\
&= \frac{\mu^2}{\theta^2}P_{11} - \frac{2}{\theta}P_{12} + \frac{1}{\mu^2}P_{22} \\
&= \wp_{22}. \qquad (3.14)
\end{aligned}$$

In the same way,

$$e(\wp_{11}) = 2\wp_{12}, \quad e(\wp_{12}) = \wp_{22}, \quad e(\wp_{22}) = 0, \qquad (3.15)$$

and

$$f(\wp_{11}) = 0, \quad f(\wp_{12}) = \wp_{11}, \quad f(\wp_{22}) = 2\wp_{12}. \qquad (3.16)$$

These $\wp_{ij} = \wp_{ji}$ still satisfy the integrability properties: $\partial_i\wp_{jk} = \partial_j\wp_{ik}$.

4. Families of identities as representations.

The $\wp$ function satisfies many interesting differential relations, of which a particularly important set is the following [2]:

$$\begin{aligned}
-\frac{1}{3}(\wp_{2222} - 6\wp_{22}^2) &= g_2g_6 - 4g_3g_5 + 3g_4^2 \\
&\qquad + g_4\wp_{22} - 2g_5\wp_{12} + g_6\wp_{11}, \\
-\frac{1}{3}(\wp_{1222} - 6\wp_{22}\wp_{12}) &= \frac{1}{2}(g_1g_6 - 3g_2g_5 + 2g_3g_4) \\
&\qquad + g_3\wp_{22} - 2g_4\wp_{12} + g_5\wp_{11}, \\
-\frac{1}{3}(\wp_{1122} - 2\wp_{22}\wp_{11} - 4\wp_{12}^2) &= \frac{1}{6}(g_0g_6 - 9g_2g_4 + 8g_3^2) \\
&\qquad + g_2\wp_{22} - 2g_3\wp_{12} + g_4\wp_{11}, \\
-\frac{1}{3}(\wp_{1112} - 6\wp_{12}\wp_{11}) &= \frac{1}{2}(g_0g_5 - 3g_1g_4 + 2g_2g_3) \\
&\qquad + g_1\wp_{22} - 2g_2\wp_{12} + g_3\wp_{11}, \\
-\frac{1}{3}(\wp_{1111} - 6\wp_{11}^2) &= g_0g_4 - 4g_1g_3 + 3g_2^2 \\
&\qquad + g_0\wp_{22} - 2g_1\wp_{12} + g_2\wp_{11}.
\end{aligned} \qquad (4.1)$$

We shall remark shortly on the connection between these equations and the Boussinesq and Korteweg-de Vries [5] equations, but for now we point out that successive application of the e operator takes us from the bottom to the topmost equation, which it annihilates, and that successive application of the f operator takes us from the top to the bottom, which it annihilates.

For example,

$$\begin{aligned}\frac{1}{3}e(\wp_{1112}-6\wp_{12}\wp_{11}) &= \frac{1}{3}(e(\wp_{1112})-6e(\wp_{12})\wp_{11}-6\wp_{12}e(\wp_{11}))\\ &= \frac{1}{3}(3\wp_{1122}-6\wp_{22}\wp_{11}-12\wp_{12}\wp_{12})\\ &= \wp_{1122}-2\wp_{22}\wp_{11}-4\wp_{12}\wp_{12},\end{aligned}$$

$$\begin{aligned}\frac{1}{3}e(g_0g_5-3g_1g_4+2g_2g_3) &= \frac{1}{3}(6g_1g_5+g_0g_6-15g_2g_4-6g_1g_5+8g_3^3+6g_2g_4)\\ &= \frac{1}{3}(g_0g_6-9g_2g_4+8g_3^2),\end{aligned}$$

and

$$\begin{aligned}\frac{1}{3}e(g_1\wp_{22}-2g_2\wp_{12}+g_3\wp_{11}) &= \frac{1}{3}(5g_2\wp_{22}-8g_3\wp_{12}-2g_2\wp_{22}+3g_4\wp_{11}+2g_3\wp_{12})\\ &= g_2\wp_{22}-2g_3\wp_{12}+g_4\wp_{11}\end{aligned}$$

This is a very simple proof of the covariance of the equations which thus form a five dimensional irreducible representation of $SL_2(\mathbb{C})$.

It further follows that we can rewrite this set of five equations as a single one by applying the vertex like operators

$$\mathcal{E}=\exp(\lambda e),\quad \mathcal{F}=\lambda^4\exp\left(\frac{1}{\lambda}f\right)$$

to either the lowest or topmost equation respectively. Application of $\mathcal{E}$ gives

$$-\frac{1}{3}\wp_{zzzz}+2\wp_{zz}^2=G(\lambda)+g_2(\lambda)\wp_{zz}-2g_1(\lambda)\wp_{z\bar{z}}+g_0(\lambda)\wp_{\bar{z}\bar{z}} \tag{4.2}$$

where the subscript z denotes the derivation $\partial_1+\lambda\partial_2$ and the subscript $\bar{z}$ denotes ∂_2. The $g_p(\lambda)$ and $G(\lambda)$ are given by $g_p(\lambda)=\mathcal{E}(g_p)$:

$$\begin{aligned}g_0(\lambda) &= g_6\lambda^6+6g_5\lambda^5+15g_4\lambda^4+20g_3\lambda^3+15g_2\lambda^2+6g_1\lambda+g_0\\ g_1(\lambda) &= g_6\lambda^5+5g_5\lambda^4+10g_4\lambda^3+10g_3\lambda^2+5g_2\lambda+g_1\\ g_2(\lambda) &= g_6\lambda^4+4g_5\lambda^3+6g_4\lambda^2+4g_3\lambda+g_2\\ g_3(\lambda) &= g_6\lambda^3+3g_5\lambda^2+3g_4\lambda+g_3\\ g_4(\lambda) &= g_6\lambda^2+2g_5\lambda+g_4\\ g_5(\lambda) &= g_6\lambda+g_5\\ g_6(\lambda) &= g_6\end{aligned}$$

and

$$\begin{aligned}G(\lambda) &= g_0(\lambda)g_4(\lambda)-4g_1(\lambda)g_3(\lambda)+3g_2(\lambda)^2\\ &= \mathcal{E}(g_0)\mathcal{E}(g_4)-4\mathcal{E}(g_1)\mathcal{E}(g_3)+3\mathcal{E}(g_2)^2,\end{aligned}$$

a polynomial, by a number of remarkable cancellations, of degree *four* only in λ.

We also have the relation (action of e),

$$g_{p+1}(\lambda)=\frac{1}{6-p}\frac{\partial g_p}{\partial\lambda}. \tag{4.3}$$

More generally, let m denote a general group element in $SL_2(\mathbb{C})$ corresponding to the transformation $x\mapsto m(x)=\frac{\alpha x+\beta}{\gamma x+\delta}$. Then define $\partial=m(\partial_1)=\alpha\partial_1-\beta\partial_2$ and

$\bar{\partial} = m(\partial_2) = -\gamma\partial_1 + \delta\partial_2$ and by summing the five equations for $\wp$ with weights α^4, $4\alpha^3\beta$, $6\alpha^2\beta^2$, $4\alpha\beta^3$ and β^4 we obtain

$$\begin{aligned} -\frac{1}{3}\partial^4\wp + 2(\partial^2\wp)^2 &= \Gamma_0\Gamma_4 - 4\Gamma_1\Gamma_3 + 3\Gamma_2^2 + \\ &\quad \Gamma_0\bar{\partial}^2\wp - 2\Gamma_1\partial\bar{\partial}\wp + \Gamma_2\partial^2\wp \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} (-\gamma x + \alpha)^6 g\left(\frac{\delta x - \beta}{-\gamma x + \alpha}\right) &= \Gamma_0 + 6\Gamma_1 x + 15\Gamma_2 x^2 + 20\Gamma_3 x^3 \\ &\quad +15\Gamma_4 x^4 + 6\Gamma_5 x^5 + \Gamma_6 x^6. \end{aligned} \tag{4.5}$$

Equation (4.4) is a family of equations parametrized by the points in $SL_2(\mathbb{C})$. We shall make use of these forms later.

The important observation then is that all relations between $\wp$-functions and between $\wp$-functions and the g_i have to be covariant, that is: they must partition themselves into sets which are permuted under the actions of e, f and h. Each such set is spanned by a finite number of relations which form a basis for a finite dimensional representation of $SL_2(\mathbb{C})$.

If we set $g_6 = 0$, $g_5 = \frac{2}{3}$, the remaining $g_i = G_i$ and $\wp = P$ we obtain the set of equations appropriate to the case where one branch point is moved to ∞:

$$\begin{aligned} -\frac{1}{3}(P_{2222} - 6P_{22}^2) &= -\frac{8}{3}G_3 + 3G_4^2 \\ &\quad +G_4P_{22} - \frac{4}{3}P_{12}, \\ -\frac{1}{3}(P_{1222} - 6P_{22}P_{12}) &= \frac{1}{2}(-2G_2 + 2G_3G_4) \\ &\quad +G_3P_{22} - 2G_4P_{12} + \frac{2}{3}P_{11}, \\ -\frac{1}{3}(P_{1122} - 2P_{22}P_{11} - 4P_{12}^2) &= \frac{1}{6}(-9G_2G_4 + 8G_3^2) \\ &\quad +G_2P_{22} - 2G_3P_{12} + G_4P_{11}, \\ -\frac{1}{3}(P_{1112} - 6P_{12}P_{11}) &= \frac{1}{2}(\frac{2}{3}G_0 - 3G_1G_4 + 2G_2G_3) \\ &\quad +G_1P_{22} - 2G_2P_{12} + G_3P_{11}, \\ -\frac{1}{3}(P_{1111} - 6P_{11}^2) &= G_0G_4 - 4G_1G_3 + 3G_2^2 \\ &\quad +G_0P_{22} - 2G_1P_{12} + G_2P_{11}. \end{aligned} \tag{4.6}$$

The residue of the $\mathfrak{sl}_2(\mathbb{C})$ action is evident in that the operator E moves us up this chain of equations, annihilating the topmost. If one relates these P back to the P^K via the subtraction of the appropriate constants one obtains the Kleinian form of these equations which is the one usually quoted [4].

The e and f operators may be used to shortcut more tedious calculations. For example, equality of cross derivatives in the set (4.1) implies identities linear in the three index symbols. Thus $\partial_1\wp_{2222} = \partial_2\wp_{1222}$, gives

$$2(\wp_{22}\wp_{122} - \wp_{12}\wp_{222}) = -g_3\wp_{222} + 3g_4\wp_{122} - 3g_5\wp_{112} + g_6\wp_{111} \tag{4.7}$$

and from this, by application of f, we obtain the four dimensional representation,

$$\begin{aligned}
2(\wp_{22}\wp_{122} - \wp_{12}\wp_{222}) &= -g_3\wp_{222} + 3g_4\wp_{122} - 3g_5\wp_{112} + g_6\wp_{111}\\
-\frac{2}{3}(\wp_{11}\wp_{222} - 2\wp_{22}\wp_{112} + \wp_{12}\wp_{122}) &= -g_2\wp_{222} + 3g_3\wp_{122} - 3g_4\wp_{112} + g_5\wp_{111}\\
\frac{2}{3}(\wp_{22}\wp_{111} - 2\wp_{11}\wp_{122} + \wp_{12}\wp_{112}) &= -g_1\wp_{222} + 3g_2\wp_{122} - 3g_3\wp_{112} + g_4\wp_{111}\\
-2(\wp_{12}\wp_{111} - \wp_{11}\wp_{112}) &= -g_0\wp_{222} + 3g_1\wp_{122} - 3g_2\wp_{112} + g_3\wp_{111}.
\end{aligned} \tag{4.8}$$

Less efficiently, these identities may be obtained by considering the other cross derivatives. Further, the corresponding identities for the Kleinian functions are obtained directly by reduction.

Being a set of four, homogeneous linear identities in four variables (the three index symbols) the equations (4.8) have to be linearly dependent which implies the vanishing of the determinant,

$$\begin{vmatrix}
g_6 & -3g_5 & 3g_4 + 2\wp_{22} & -g_3 - 2\wp_{12}\\
-3g_5 & 9g_4 - 4\wp_{22} & -9g_3 + 2\wp_{12} & 3g_2 + 2\wp_{11}\\
3g_4 + 2\wp_{22} & -9g_3 + 2\wp_{12} & 9g_2 - 4\wp_{11} & -3g_1\\
-g_3 - 2\wp_{12} & 3g_2 + 2\wp_{11} & -3g_1 & g_0
\end{vmatrix} = 0. \tag{4.9}$$

This must either be identically true or the expression for the Kummer surface in the case of *generic* coefficients of the sextic. In fact it is the latter [2]. Expanding the determinant leads to a rather complex equation which breaks up into five parts of degrees 0, 1, 2, 3 and 4 in the $\wp_{ij}$, each of which is an invariant under the $SL_2(\mathbb{C})$ action. The leading order term is the invariant $16(\wp_{12}^2 - \wp_{22}\wp_{11})^2$.

Again one must reduce by the usual procedure to recover the Kleinian form (1.4).

In order to obtain the classical identities for quadratics in three index symbols, consider

$$\begin{aligned}
\partial_2(\wp_{222}^2) &= 2\wp_{222}\wp_{2222}\\
&= 2\wp_{222}(6\wp_{22}^2 - 3(g_2g_6 - 4g_3g_5 + 3g_4^2) - 3g_4\wp_{22} + 6g_5\wp_{12} - 3g_6\wp_{11})\\
&= \partial_2\big(4\wp_{22}^3 - 6(g_2g_6 - 4g_3g_5 + 3g_4^2)\wp_{22} - 3g_4\wp_{22}^2\big)\\
&\quad -6(g_6\wp_{11} - 2g_5\wp_{12})\wp_{222}.
\end{aligned} \tag{4.10}$$

Elimination of $\wp_{111}$ between the first two of equations (4.8) yields,

$$\begin{aligned}
&2g_5(\wp_{22}\wp_{122} - \wp_{12}\wp_{222}) + \frac{2}{3}g_6(\wp_{11}\wp_{222} - 2\wp_{22}\wp_{112} + \wp_{12}\wp_{122})\\
= \;& -\partial_2\{(g_5g_3 - g_6g_2)\wp_{22} - 3(g_5g_4 - g_3g_6)\wp_{12} + 3(g_5^2 - g_6g_4)\wp_{11}\}
\end{aligned}$$

which can be rewritten

$$\begin{aligned}
2(g_6\wp_{11} - 2g_5\wp_{12})\wp_{222} &= \partial_2\{(-(g_5g_3 - g_6g_2)\wp_{22} + 3(g_5g_4 - g_3g_6)\wp_{12}\\
&\quad -3(g_5^2 - g_6g_4)\wp_{11} - 2g_5\wp_{22}\wp_{12} - \frac{1}{3}g_6(\wp_{12}^2 - 4\wp_{11}\wp_{22})\},
\end{aligned}$$

thus allowing the right hand side of (4.10) to be written as a total ∂_2 derivative.

Integrating,

$$\begin{aligned}\wp_{222}^2 &= 4\wp_{22}^3 - 3g_4\wp_{22}^2 + 6g_5\wp_{12}\wp_{22} + g_6\wp_{12}^2 - 4g_6\wp_{11}\wp_{22}\\ &+9(g_5^2 - g_4g_6)\wp_{11} + 9(g_6g_3 - g_4g_5)\wp_{12}\\ &+9(3g_3g_5 - g_2g_6 - 2g_4^2)\wp_{22} + C_6.\end{aligned}$$

Here C_6 is a constant function of the g_i which must be the highest weight for a seven dimensional representation, $\{C_6, C_5, C_4, C_3, C_2, C_1, C_0\}$ of $SL_2(\mathbb{C})$ so that application of f to the above creates the seven identities

(4.11)
$$\begin{aligned}\wp_{222}^2 = 4\wp_{22}^3 &- 3g_4\wp_{22}^2 + 6g_5\wp_{12}\wp_{22} + g_6\wp_{12}^2 - 4g_6\wp_{11}\wp_{22}\\ &+ 9(g_5^2 - g_4g_6)\wp_{11} - 9(g_4g_5 - g_6g_3)\wp_{12}\\ &+ 9(3g_3g_5 - g_2g_6 - 2g_4^2)\wp_{22} + C_6,\end{aligned}$$

(4.12)
$$\begin{aligned}6\wp_{122}\wp_{222} &= 24\wp_{12}\wp_{22}^2 + 18g_5\wp_{12}^2 - 12g_3\wp_{22}^2 + 18g_4\wp_{12}\wp_{22}\\ &- 6g_6\wp_{12}\wp_{11} - 18g_5\wp_{11}\wp_{22} + 27(g_4g_5 - g_6g_3)\wp_{11}\\ &+ 9(g_2g_6 - 9g_4^2 + 8g_3g_5)\wp_{12} + 9(3g_2g_5 - g_3g_4 - 2g_1g_6)\wp_{22} + C_5\end{aligned}$$

(4.13)
$$\begin{aligned}9\wp_{122}^2 &= 48\wp_{12}^2\wp_{22} + 12\wp_{11}\wp_{22}^2 + 12g_3\wp_{12}\wp_{22} + 63g_4\wp_{12}^2 - 36g_4\wp_{11}\wp_{22}\\ &+ 6\wp_{112}\wp_{222} - 3g_6\wp_{11}^2 - 18g_5\wp_{12}\wp_{11} - 18g_2\wp_{22}^2\\ &+ 9(g_3g_5 - 4g_2g_6 + 3g_4^2)\wp_{11} + 9(18g_2g_5 - 17g_3g_4 - g_1g_6)\wp_{12}\\ &- 9(3g_1g_5 + 2g_3^2 - 6g_2g_4 + g_0g_6)\wp_{22} + C_4\end{aligned}$$

(4.14)
$$\begin{aligned}18\wp_{122}\wp_{112} &= 48\wp_{12}\wp_{11}\wp_{22} + 32\wp_{12}^3 - 12g_5\wp_{11}^2 + 92g_3\wp_{12}^2\\ &+ 2\wp_{111}\wp_{222} - 12g_4\wp_{12}\wp_{11} - 12g_2\wp_{12}\wp_{22} + 48g_3\wp_{11}\wp_{22} - 12g_1\wp_{22}^2\\ &- 9(3g_1g_6 - 4g_3g_4 + g_2g_5)\wp_{11} + 9(17g_2g_4 - 18g_3^2 - g_0g_6 + 8g_1g_5)\wp_{12}\\ &- 9(g_1g_4 - 4g_2g_3 + 3g_0g_5)\wp_{22} + C_3\end{aligned}$$

(4.15)
$$\begin{aligned}6\wp_{122}\wp_{111} &= 48\wp_{12}^2\wp_{11} + 12\wp_{11}^2\wp_{22} + 12g_3\wp_{12}\wp_{11} + 63g_2\wp_{12}^2\\ &+ 9\wp_{112}^2 - 18g_4\wp_{11}^2 - 3g_0\wp_{22}^2 - 36g_2\wp_{11}\wp_{22} - 18g_1\wp_{12}\wp_{22}\\ &- 9(g_0g_6 + 3g_1g_5 - 6g_2g_4 + 2g_3^2)\wp_{11} - 9(17g_2g_3 + g_0g_5 - 18g_4g_1)\wp_{12}\\ &- 9(4g_0g_4 - g_1g_3 - 3g_2^2)\wp_{22} + C_2\end{aligned}$$

(4.16)
$$\begin{aligned}6\wp_{112}\wp_{111} &= 24\wp_{12}\wp_{11}^2 + 18g_2\wp_{12}\wp_{11} + 18g_1\wp_{12}^2 - 12g_3\wp_{11}^2\\ &- 6g_0\wp_{12}\wp_{22} - 18g_1\wp_{11}\wp_{22} - 9(2g_5g_0 + g_2g_3 - 3g_4g_1)\wp_{11}\\ &+ 9(g_4g_0 - 9g_2^2 + 8g_3g_1)\wp_{12} - 27(g_0g_3 - g_1g_2)\wp_{22} + C_1\end{aligned}$$

(4.17)
$$\begin{aligned}\wp_{111}^2 = 4\wp_{11}^3 &+ 6g_1\wp_{12}\wp_{11} + g_0\wp_{12}^2 - 3g_2\wp_{11}^2 - 4g_0\wp_{11}\wp_{22}\\ &- 9(g_4g_0 + 2g_2^2 - 3g_3g_1)\wp_{11}\\ &- 9(g_2g_1 - g_3g_0)\wp_{12} + 9(g_1^2 - g_0g_2)\wp_{22} + C_0\end{aligned}$$

These quadratic relations are, like the expression for the Kummer surface earlier, valid for the branch points of the curve in general position.

The constant C_0 can be identified by going to the canonical form, using the associated definitions of the P_{ij} and expanding in the independent variables X_1 and X_2 about $(0, 0)$ (assuming $g - 0 \neq 0$) in the last of the above equations. Because g_5 and g_6 do not feature in this equation its form is retained when the branch point is moved to infinity. The lowest order (constant) terms in P_{11} etc. are

$$\begin{aligned} P_{11} &\approx -\frac{9}{4}\frac{g_2g_0 - g_1^2}{g_0} \\ P_{12} &\approx \frac{1}{2}g_3 \\ P_{22} &\approx \frac{3}{2}g_4 \\ P_{111} &\approx -\frac{1}{4}\frac{20g_3g_0^2 - 45g_1g_2g_0 + 27g_1^3}{g_0^{3/2}}, \end{aligned}$$

which, when substituted into the last equation yield

$$C_0 = \frac{81}{4}(g_3^2g_0 - 2g_3g_1g_2 + g_2^3 + g_1^2g_4 - g_0g_2g_4). \tag{4.18}$$

It is easily checked that $f(C_0) = 0$ and we generate the other C_i by applying e:

$$\begin{aligned} C_1 = e(C_0) &= \frac{81}{2}(-g_1g_3^2 - g_1g_2g_4 + g_3g_2^2 + g_3g_0g_4 + g_5g_1^2 - g_5g_2g_0) \\ C_2 = \frac{1}{2}e(C_1) &= \frac{81}{4}(-4g_1g_3g_4 + 2g_1g_5g_2 + 3g_2g_3^2 - 2g_2^2g_4 - 2g_3g_5g_0 + 3g_0g_4^2 + \\ &\quad + g_6g_1^2 - g_6g_2g_0) \\ C_3 = \frac{1}{3}e(C_2) &= \frac{81}{2}(-2g_1g_3g_5 + g_1g_4^2 + g_1g_6g_2 - 3g_2g_3g_4 + g_5g_2^2 + 2g_3^3 - g_3g_6g_0 + \\ &\quad + g_4g_5g_0) \\ C_4 = \frac{1}{4}e(C_3) &= \frac{81}{4}(-2g_1g_3g_6 + 2g_1g_4g_5 - 4g_2g_3g_5 - 2g_2g_4^2 + 3g_6g_2^2 + 3g_3^2g_4 - \\ &\quad - g_4g_6g_0 + g_5^2g_0) \\ C_5 = \frac{1}{5}e(C_4) &= \frac{81}{2}(-g_1g_4g_6 + g_1g_5^2 + g_2g_3g_6 - g_2g_4g_5 - g_3^2g_5 + g_3g_4^2) \\ C_6 = \frac{1}{6}e(C_5) &= \frac{81}{4}(-g_2g_4g_6 + g_2g_5^2 + g_3^2g_6 - 2g_3g_4g_5 + g_4^3) \end{aligned}$$

Let us remark in passing that the Klein formula [4]

$$P_{11}^K + (X_1 + X_2)P_{12}^K + X_1X_2P_{22}^K = \frac{F(X_1, X_2) - 2Y_1Y_2}{4(X_1 - X_2)^2}. \tag{4.19}$$

(usually written with the symbols $\wp_{ij}$) is not, of course, respected by the E action for the reasons already stated. However, if we modify the polar form appropriately to

$$\hat{F}(X_1, X_2) = F(X_1, X_2) + 2(X_1 - X_2)^2(3G_4X_1X_2 + G_3(X_1 + X_2) + 3G_2), \tag{4.20}$$

then the *modified* Klein formula

$$P_{11} + (X_1 + X_2)P_{12} + X_1X_2P_{22} = \frac{\hat{F}(X_1, X_2) - 2Y_1Y_2}{4(X_1 - X_2)^2} \tag{4.21}$$

is annihilated by E. Indeed, more than this, the expression

$$\wp_{11} + (x_1 + x_2)\wp_{12} + x_1 x_2 \wp_{22} = \frac{\hat{F}(x_1, x_2) - 2y_1 y_2}{4(x_1 - x_2)^2}, \tag{4.22}$$

that is, the variables all being in generic position, and $\hat{F}$ being formed with the generic values of the g_i, is actually covariant under both e and f. After substitution for the $\wp_{ij}$ in terms of the P_{ij} the left hand side takes the form

$$\frac{x_1 x_2}{\mu^2 X_1 X_2}(P_{11} + (X_1 + X_2)P_{12} + X_1 X_2 P_{22})$$

and the verification of formula (4.22) reduces to that of the identity

$$\hat{F}\left(-\frac{\theta x_1}{\mu^2(x_1 - \theta)}, -\frac{\theta x_2}{\mu^2(x_2 - \theta)}\right)\left(\frac{\mu}{\theta}\right)^6 (x_1 - \theta)^3 (x_2 - \theta)^3 = \hat{F}(x_1, x_2)$$

which is easily seen to be true. Formula (4.22) is to be found in Baker [2].

5. The Boussinesq connection and the reduction to KdV

For recent work on the Boussinesq equation see [3, 8], and references therein.

It has been remarked elsewhere [4, 7] that the first of equations (1.3), if differentiated with respect to U_2 and expressed in terms of $\phi = P^K_{22}$ becomes the KdV equation

$$\phi_{222} - 12\phi\phi_2 = 15G_4\phi_2 + 4\phi_1 \tag{5.1}$$

under the identification of U_1 with the time and U_2 with the space variable. (The G_4 term is removable by a Galilean boost.) But it does not appear to have been noted before that the system is similarly related to the Boussinesq equation. Specifically, differentiation of the last of the equations twice with respect to U_1 and putting $\psi = P^K_{11}$ yields

$$\psi_{111} - 12\psi_1^2 - 12\psi\psi_{11} = -3G_0\psi_{22} + 6G_1\psi_{12} + 15G_2\psi_{11}. \tag{5.2}$$

Again the ψ_{12} term can be removed with a boost and Boussinesq emerges when U_2 is identified with time and U_1 with space (the reverse identification to that for the KdV).

However, this relation goes deeper when it is recognised that the whole λ dependent family (4.2) is of Boussinesq form and, further, that it reduces to the KdV equation (with the same identification of space/time variables) precisely when the parameter λ is a root of the sextic $g_0(\lambda) = g(\lambda) = 0$. Of course, whilst we have the Boussinesq equation for any particular choice of λ, the full set of equations (equivalently the λ-family) are a far stronger constraint.

These remarks also apply to the equation on the whole group (4.4) when α and β are such that $\Gamma_0 = 0$. Being an integrable system, equations (4.2) and (4.4) are the compatibility conditions of pairs of Lax operators. These Lax operators will be sections of the tangent bundle over the Jacobian of the genus 2 curve.

6. The Lax pair for Baker's equations.

The Lax operators for the λ-family of Boussinesq equations (4.2) are

$$\begin{aligned} L(\lambda) &= \zeta\partial_{\bar{z}} + \partial_z^2 - 2\wp_{zz} \\ M(\lambda) &= \partial_z^3 + \frac{1}{2}\zeta'\partial_z^2 + \frac{1}{20}(\zeta\zeta'' + \zeta'^2)\partial_z - 3\wp_{zz}\partial_z - \frac{3}{2}\wp_{zzz} - \zeta'\wp_{zz} + \frac{3}{2}\zeta\wp_{z\bar{z}} \end{aligned} \tag{6.1}$$

where $\zeta^2 = g(\lambda)$ and prime denotes derivation with respect to λ. For each λ, $L(\lambda)$ and $M(\lambda)$ are commuting operators on the Jacobian variety. They have analytic expansions about $\lambda = 0$ (assumed a regular point) of the forms

$$L(\lambda) = \mathcal{E}(L_0) = \sum_{p=0}^{\infty} \frac{\lambda^p}{p!} L_p, \tag{6.2}$$

$$M(\lambda) = \mathcal{E}(M_0) = \sum_{p=0}^{\infty} \frac{\lambda^p}{p!} M_p, \tag{6.3}$$

where $L_{p+1} = e(L_p)$, $M_{p+1} = e(M_p)$ and

$$L_0 = g_0^{\frac{1}{2}} \partial_2 + \partial_1^2 - 2\wp_{11}, \tag{6.4}$$

$$\begin{aligned} M_0 &= \partial_1^3 + \frac{3}{2} g_0^{-\frac{1}{2}} g_1 \partial_1^2 + \left(\frac{3}{4} g_2 - 3\wp_{11}\right) \partial_1 \\ &\quad -\frac{3}{2}\wp_{111} - 3 g_0^{-\frac{1}{2}} g_1 \wp_{11} + \frac{3}{2} g_0^{\frac{1}{2}} \wp_{12}. \end{aligned} \tag{6.5}$$

Straightforward application of e yields

$$L_1 = 3 g_0^{-\frac{1}{2}} g_1 \partial_2 + 2\partial_1\partial_2 - 4\wp_{12} \tag{6.6}$$

$$L_2 = (15 g_0^{-\frac{1}{2}} g_2 - 9 g_0^{-\frac{3}{2}} g_1^2)\partial_2 + 2\partial_2^2 - 4\wp_{22} \tag{6.7}$$

$$L_p = k_p \partial_2 \quad p > 2, \tag{6.8}$$

where the k_p are constant functions of $g_0, \ldots g_6$ only.

Application of e to M_0 yields (more involved) expressions for the M_p.

The commutation conditions also expand in an analytic series in λ:

$$\begin{aligned} & [L_0, M_0] = 0 \\ e([L_0, M_0]) &= [L_1, M_0] + [L_0, M_1] = 0 \\ e^2([L_0, M_0]) &= [L_2, M_0] + 2[L_1, M_1] + [L_0, M_2] = 0 \\ & \text{etc.} \end{aligned}$$

The first five of these relations generate the Baker equations. All others are identically zero. We can, of course, summarise these in a conventional matrix Lax pair

$$[\mathbb{L}, \mathbb{M}] = 0, \tag{6.9}$$

with

$$\mathbb{L} = \begin{pmatrix} L_0 & L_1 & \frac{1}{2}L_2 & \frac{1}{6}L_3 & \frac{1}{24}L_4 \\ 0 & L_0 & L_1 & \frac{1}{2}L_2 & \frac{1}{6}L_3 \\ 0 & 0 & L_0 & L_1 & \frac{1}{2}L_2 \\ 0 & 0 & 0 & L_0 & L_1 \\ 0 & 0 & 0 & 0 & L_0 \end{pmatrix},$$

and

$$\mathbb{M} = \begin{pmatrix} M_0 & M_1 & \frac{1}{2}M_2 & \frac{1}{6}M_3 & \frac{1}{24}M_4 \\ 0 & M_0 & M_1 & \frac{1}{2}M_2 & \frac{1}{6}M_3 \\ 0 & 0 & M_0 & M_1 & \frac{1}{2}M_2 \\ 0 & 0 & 0 & M_0 & M_1 \\ 0 & 0 & 0 & 0 & M_0 \end{pmatrix}.$$

7. A family of solutions to Boussinesq.

Finally, it follows from our representation theoretic treatment of the Baker equations that the genus two $\wp$ function does indeed provide a family of solutions to the Boussinesq equation. We can describe this family explicitly using the following argument.

Let $\wp$ be associated with the curve $y^2 = g(x)$ in the classical manner. It will satisfy the last of equations (4.1) in particular. Applying ∂_1^2 and putting $u = -12\partial_1^2\wp$ simplifies the coefficients to give

$$\partial_1^4 u + u\partial_1^2 u + (\partial_1 u)^2 + g_2\partial_1^2 u - 2g_1\partial_1\partial_2 u + g_0\partial_2^2 u = 0.$$

Replacing the derivatives by

$$\partial_2 = g_0^{-\frac{1}{2}}\partial_T + \frac{g_1}{g_0}\partial_X \tag{7.1}$$

$$\partial_1 = \partial_X \tag{7.2}$$

and putting

$$u = w - \frac{g_0 g_2 - g_1^2}{g_0}$$

leaves us with the Boussinesq equation for w,

$$w_{XXXX} + w w_{XX} + w_X^2 + w_{TT} = 0. \tag{7.3}$$

Undoing these changes gives the expression for a family of solutions:

$$w(X,T) = \frac{g_0 g_2 - g_1^2}{g_0} - 12\partial_X^2\wp(X - g_0^{-\frac{3}{2}} g_1 T, g_0^{\frac{1}{2}} T). \tag{7.4}$$

Here $\wp(u_1, u_2)$ is just the $\wp$ function associated with the curve with coefficients $g_0, \ldots, g_6$, whose arguments are the canonical variables u_1 and u_2 on the Jacobian variety. The function $w(X,T)$ will also satisfy the other four of the equations (4.1) and is thus not a general solution to Boussinesq.

8. Conclusions and comments

We have shown how the covariance property of the underlying family of algebraic curves provides a new tool for the study of identities between classical $\wp$-functions. In particular we have used a connection with the well-known Boussinesq equation to derive Lax operators for the Baker equations and to examine their covariance. We have described a family of solutions to the Boussinesq equation in terms of the genus two $\wp$ function.

One important function of this paper has been to modernise the treatment of $\wp$ functions given in Baker's book [2].

In a separate publication we present a reformulation of the theory in which this covariance takes centre stage. Our hope is that these methods will enormously simplify the treatment of curves of higher genus.

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